The Bounds of Logic

A Generalized Viewpoint

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For Peter

Preface

Whatever the fate of the particulars, one thing is certain. There is no going back to the view that logic is [standard] first-order logic.
Jon Barwise, *Model-Theoretic Logics*

When I went to Columbia University to study with Prof. Charles Parsons, I felt I was given a unique opportunity to work on "foundational" issues in logic. I was interested not so much in the controversies involving logicism, intuitionism, and formalism as in the ideas behind "core" logic: first-order Fregean, Russellian, Tarskian logic. I wanted to understand the philosophical force of logic, and I wanted to approach logic critically.

Philosophical investigations of logic are difficult in that a fruitful point of view is hard to find. My own explorations started off when Prof. Parsons pointed out to me that some mathematicians and linguists had generalized the standard quantifiers. Generalization of quantifiers was something I was looking for since coming upon Quine's principle of ontological commitment. If we understood the universal and existential quantifiers as particular instances of a more general form, perhaps we would be able to judge whether quantification carries ontological commitment. So the idea of generalized quantifiers had an immediate appeal, and I sat down to study the literature.

The generalization of quantifiers gives rise to the question, What is logic? in a new, sharp form. In fact, it raises two questions, mutually stimulating, mutually dependent. More narrowly, these questions concern quantifiers, but a broader outlook shifts the emphasis: What is it for a *term* to be logical? What are *all* the terms of logic? Sometimes in the course of applying a principle, we acquire our deepest understanding of it, and in the attempt to extend a theory, we discover what drives it. In this vein I thought that to determine the full scope of logical terms, we have to understand the idea of logicality. But the actual expansion of quantifiers gives us hands-on experience that is, in turn, valuable in tackling "logicality." Prof. Parsons encouraged me to select this as the topic of my dissertation, and *The Bounds of Logic*, a revised version of my thesis, follows the course of my inquiries.

The idea of logic with "generalized" quantifiers has, in the last decades, commanded the attention of mathematicians, philosophers, linguists, and cognitive scientists. My own perspective is less abstract than that of most mathematicians and less empirical than the viewpoint of linguists and cognitive scientists. I decided not to address the logical structure of natural language directly. Instead, I would follow my philosophical line of reasoning unmitigatedly and then see how the theory fared in the face of empirical data. If the reasoning was solid, the theory would have a fair chance of converging with a sound linguistic theory, but as a philosophical outlook, it should stand on its own.

The book grew out of three papers I wrote between 1984 and 1987: "First-Order Quantifiers and Natural Language" (1984), "Branching Quantifiers, First-Order Logic, and Natural Language" (1985), and "Logical Terms: A Semantic Point of View" (1987). These provide the backbone of chapters 2, 5, and 3. Chapter 1 is based on my thesis proposal, and the ideas for chapter 4 were formulated soon after the proposal defense. "Logical Terms" was rewritten as "A Conception of Tarskian Logic" and supplied with a new concluding section. This section, with slight variations, constitutes chapter 6. An abridged version of "A Conception of Tarskian Logic" appeared in *Pacific Philosophical Quarterly* (1989a). I would like to thank the publishers for their permission to reproduce extensive sections of the paper. "Branching Quantifiers" gave way to "Ways of Branching Quantifiers" and was published in *Linguistics and Philosophy* (1990b). I am thankful to the publishers of this journal for allowing me to include the paper (with minor revisions) here.

At the same time that I was working on my thesis, other philosophers and semanticists were tackling tangential problems. In general, my guideline was to follow only those leads that were directly relevant. A few related essays appeared too late to affect my inquiry. In the final revision I added some new references, but for the most part I did not change the text. I felt that the original conception of the book had the advantage of naturally leading the reader from the questions and gropings of the early chapters to the answers in the middle and from there to the formal developments and the philosophical ending.

There is, however, one essay that I would like to mention here because it is so close to mine in its spirit and its view on the scope of logic. This is Dag Westerståhl's unpublished dissertation, "Some Philosophical Aspects of Abstract Model Theory" (1976), which I learned of a short time before the final revision of this book was completed. Had I come upon it in the early stages of my study, I am sure it would have been a source of inspiration and an influence upon my work. As it turned out, Westerståhl relied on a series of "intuitions about logic," while I set out to investigate the bounds of logic as a function of its goal, drawing upon Tarski's early writings on the foundations of semantics.

Chapter by chapter, I proceed as follows: In chapter 1, I set down the issues the book attempts to resolve and I give an outline of my philosophical approach to logic. Chapter 2 analyzes Mostowski's original generalization of quantifiers, tracing its roots to Frege's conception of statements of number. The question then arises of how to extend Mostowski's work. I discuss a proposal by Barwise and Cooper (1981) to create a system of *nonlogical* quantifiers for use in linguistic representation. Pointing to weaknesses in Barwise and Cooper's approach, I advocate in its place a straightforward extension of the *logical* quantifiers, as in Lindström (1966a), and show how this can be naturally applied in natural-language semantics. It is not clear, however, what the philosophical principle behind Mostowski's work is. To determine the scope of logical quantifiers in complete generality, we need to analyze the notion of "logicality." This leads to chapter 3.

For a long time I thought I would not be able to answer the questions posed in this work. I would present the issues in a sharp and, I hoped, stimulating form, but as for the answers, I had no idea what the guiding principle should be. How would I know whether a given term, say "being a well-ordering relation," is a logical term or not? What criterion could be used as an objective arbiter? The turning point for me was John Etchemendy's provocative essay on Tarski. Etchemendy's charge that Tarski committed a simple fallacy sent me back to the old papers, and words that were too familiar to convey a new meaning suddenly came to life. My answer to the question of logicality has three sides: First, it is an analysis of the ideas that led Tarski to the construction of the syntactic-semantic system that has been a paradigm of logic ever since. Second, it is an argument for the view that the original ideas were not fully realized by the standard system; it takes a far broader logical network to bring the Tarskian project to true completion. Finally, the very principles that underlie modern semantics point the way to a simple, straightforward criterion of logicality. I spell out this criterion and I discuss the conception of logic that ensues. As a side note I should say that although chapter 3 was not written as a defense of the historical Tarski, it contains, I believe, all that is needed to prove the consistency of Tarski's approach.

Chapter 4 presents a formal semantics for the "unrestricted" first-order logic whose boundaries were delineated in chapter 3. The semantic system is essentially coextensional with Lindström's, but the method of definition is constructive—a semantics "from the ground up." What I try to show, first informally and later formally, is how we can build the logical terms over a given universe by starting with individuals and constructing the relations and predicates that will form the extensions of logical terms over that universe. Chapter 4 also investigates the enrichment of logical vocabulary as a tool in linguistic semantics, pointing to numerous applications and showing how increasingly "stronger" quantifiers are required for certain complex constructions.

Chapter 5 was the most difficult chapter to write. Whereas the book in general investigates the scope and limits of logical "particles," this chapter inquires into new possibilities of combining particles together. My original intent was to study the new, "branching" structure of quantifiers to determine whether it belongs to the new conception of logic. But upon reading the literature I found that in the context of "generalized" logic it has never been determined what the branching structure really is. Jon Barwise's pioneer work pointed to several partial answers, but a general semantics for branching quantifiers had yet to be worked out. My search for the branching principle led to a new, broader account than was given in earlier writings. I introduce a simple first-order notion of branching, "independence"; I universalize the existent definitions due to Barwise; and I point to a "family" of branching structures that include, in addition to "independent," Henkin, and Barwise quantifiers, also a whole new array of logico-linguistic quantifier constructions.

In chapter 6, I draw several philosophical consequences of the view of logic developed earlier in the book. I discuss the role of mathematics in logic and the metaphysical underpinning of semantics, I investigate the impact of the new conception of logic on the logicist thesis and on Quine's ontological-commitment thesis, and I end with a proof-theoretical outlook. This chapter is both a summation and, I hope, an opening for further philosophical inquiries.

The bounds of logic, on my view, are the bounds of mathematical reasoning. Any higher-order mathematical predicate or relation can function as a logical term, provided it is introduced in the right way into the syntactic-semantic apparatus of first-order logic. Logic provides a special framework for formalizing theories, a framework that draws out their necessary and formal consequences. Every formal and necessary consequence is identified by some logic, and only necessary and formal consequences pass the test of logicality. This view is accepted in practice by many logicians working in "abstract" first-order logic. My view also stands in basic agreement with that of natural-language logicians. Extended logic has made a notable contribution to linguistic analysis. Yet logical form in linguistics is often constrained by conditions that have no bearing on philosophy. To my mind, this situation is natural and has no limiting effect on the scope of logic.

On the other side of the mat stand two approaches to logic: First and obviously, there is the traditional approach, according to which standard logic is the whole of logic. No more need be said about this view. But from another direction some philosophers see in the collapse of traditional logic a collapse of logic itself as a distinctive discipline. With this view I adamantly disagree. Logic is broader than traditionally thought, but that does not mean anything goes. The boundaries of logic are based on a sharp, natural distinction. This distinction serves an important methodological function: it enables us to recognize a special type of consequence. To relinquish this distinction is to give up an important tool for the construction and criticism of theories.

The writing of this work was a happy experience, and I am very thankful to teachers, colleagues, friends, and family who helped me along the way.

I was very fortunate to work with Charles Parsons throughout my years at Columbia University. His teaching, his criticism, the opportunity he always gave me to defend my views, his expectation that I tackle problems I was not sure I could solve—all were invaluable not only for this book but for the development of my philosophical thought. I am most grateful to him.

My first dissertation committee was especially supportive and enthusiastic, and I would like to thank Robert May and Wilfried Sieg for this and for their continuing interest in my work after they left Columbia. Robert May was actively involved with my book until its completion, and I am very thankful to him for his constructive remarks and for urging me to explore the linguistic aspect of logic. Isaac Levi and Shaugan Lavine joined my dissertation committee at later stages. Levi taught me at Columbia, and his ideas had an impact on my thought. I thank him for this and

for his conversation and support. Shaughan Lavine contributed numerous useful comments on my thesis, and I am very thankful to him.

John Etchemendy sent me his works on Tarski. These were most important in developing my own views, and I am grateful to him.

During the academic year 1987/1988 I was a visiting scholar at MIT, and I would like to thank the Department of Linguistics and Philosophy for its hospitality. I had interesting and stimulating conversations with George Boolos, Jim Higginbotham, Richard Larson, and Noam Chomsky, and I am particularly thankful to Richard Cartwright for his contribution to my understanding of Tarski.

While writing the dissertation I was teaching first at Queens College and later at Barnard College. I would like to thank the members of the two philosophy departments for the supportive environment. To Alex Orenstein, Sue Larson, Hide Ishiguro, Robert Tragesser, and Palle Yourgrau I am thankful for their conversation and friendliness.

The thesis developed into a book while I was at my present position at the University of California, San Diego. I am very grateful to my new friends and colleagues at UCSD for the stimulating and friendly atmosphere. I am especially indebted to Philip Kitcher for his conversation and advice. I am also very thankful to Oron Shagrir for preparation of the indexes.

Betty Stanton of Bradford Books, The MIT Press, encouraged me to orient the book to a wider audience than I envisaged earlier. I am very thankful for her suggestions.

As editor of *Linguistics and Philosophy* Johan van Benthem commented on my "Ways of Branching Quantifiers," and his comments, as well as those of two anonymous referees, led to improvements that were carried over to the book. I appreciate these comments. I am also thankful for comments by referees of The MIT Press.

Hackett Publishing Company allowed me to cite from Tarski's works. I am thankful for their permission.

I gave several talks on branching quantifiers, and I would like to thank the audiences at the Linguistic Institute (1986), MIT (1987), and the University of Texas at Austin (1990).

My interest in the philosophy of logic arose when I was studying philosophy at the Hebrew University of Jerusalem. I am grateful to my teachers there, especially Eddy Zemach and Dale Gottlieb, whose stimulating discussions induced my active involvement with issues that eventually led to my present work.

I could not have written this book without the abounding support given to me by my family. My sons, Itai and Shlomi, were very young when I started by studies at Columbia University, and they grew up with my work. I thank you, Itai and Shlomi, for your patience with a busy mother, for your liveliness and philosophical thoughts, for rooting for me, and for much much more. My mother took over many of my duties during the last months of preparing the book. I am most grateful to her for this and for her support throughout my studies.

It is too late to thank my father, Shlomo Yoffe. He was a staunch rationalist who believed in straight, honest reasoning. He taught me to think and enjoy thinking. He respected intellectuals but not academic titles. His influence, more than anyone else's, led to my career in philosophy.

I am dedicating this book to my husband, Peter Sher, with love and gratitude. He followed my work closely from its beginning, and his suggestions regarding style and presentation of ideas left their marks on every page of this book. With great generosity he encouraged me to resume my studies in philosophy some time ago, and with great generosity he has supported me in my work since then.

The Bounds of Logic

Chapter 1
New Bounds?

"Logic," Russell said, "consists of two parts. The first part investigates what propositions are and what forms they may have. . . . The second part consists of certain supremely general propositions which assert the truth of all propositions of certain forms. . . . The first part . . . is the more difficult, and philosophically the more important; and it is the recent progress in this part, more than anything else, that has rendered a truly scientific discussion of many philosophical problems possible."[1]

The question underlying this work is, Are generalized quantifiers a case in question? Do they give rise to new, philosophically significant logical forms of propositions "enlarging our abstract imagination, and providing . . . [new] possible hypotheses to be applied in the analysis of any complex fact"?[2] Does the advent of generalized quantifiers mark a genuine breakthrough in modern logic? Has logic, in Russell's turn of expression, given thought new wings once again?

Generalized quantifiers were first introduced as a "natural generalization of the logical quantifiers" by A. Mostowski in his 1957 paper "On a Generalization of Quantifiers"[3]. Mostowski conceived his generalized quantifiers semantically as functions from sets of objects in the universe of a model for first-order logic to the set of truth values, {truth, falsity}, and syntactically as first-order formula-building operators that, like the existential and universal quantifiers, bind well-formed formulas with individual variables to form other, more complex well-formed formulas. Mostowski's quantifiers acquired the name "cardinality quantifiers," and some typical examples of these are "there are finitely many *x* such that . . . ," "most things *x* are such that . . . ," etc.

Mostowski's paper opened up the discussion of generalized quantifiers in two contexts. The first and more general context is that of the scope and subject matter of logic. Although Mostowski declared that at least some generalized quantifiers belong in any systematic presentation of symbolic logic, this aspect, with the foundational issues it raises, was not thoroughly investigated either by him or by other mathematicians who took up the subject. The second, more specific context has to do with the properties of formal first-order systems with generalized quantifiers, particularly in comparison to "classical" first-order logic and its characteristic properties: completeness, compactness, the Löwenheim-Skolem property, etc. This was the main concern of Mostowski's research, and it became the focus of the ensuing surge of mathematical interest in the subject.[4]

In contrast to the extensive and prolific treatment that generalized quantifiers have received in mathematics, the philosophical yield has been rather sparse. The philosophical significance of generalized quantifiers was examined in a small number of contemporary papers by such authors as L. H. Tharp (1975), C. Peacocke (1976), I Hacking (1979), T. McCarthy (1981) and G. Boolos (1984b) as part of an attempt to provide a general characterization of logic and logical constants. The mathematical descriptions of generalized quantifiers and the numerous constructions by mathematicians of first-order systems with new quantifiers prompted the question of whether such quantifiers are genuinely logical. Although the discussions mentioned above are illuminating, they reach no definite or compelling conclusions, and, to the best of my knowledge and judgement, the question is still open.

To inquire whether "generalized" quantifiers are logical in complete generality, we have to ascend to a conceptual linguistic scheme that is independent of, or prior to, the determination of logical constants and, in particular, logical quantifiers. We will then be able to ask, What expressions in that linguistic scheme are logical quantifiers? What are all its logical quantifiers? The scheme has to be comprehensive enough, of course, to suit the general nature of the query.

A conceptual scheme like Frege's hierarchy of levels naturally suggests itself. In such a scheme the level of a linguistic expression can be determined prior to, and independently of, the determination of its status as a logical or nonlogical expression.[5] And the principles underlying the hierarchy—namely the characterization of expressions as complete or incomplete and the classification of the latter according to the number and type of expressions that can complete them—are universally applicable. From the point of view of Frege's hierarchy of levels, the system of standard first-order logic consists of a first-level language plus certain second-level unary predicates (i.e., the universal and/or existential quantifiers) and a "complete" set of truth-functional connectives. Our question can now be formulated with respect to Frege's linguistic scheme as follows: Which second-level predicates and relations can combine with first-level predicates, relations, functional expressions, proper names and sentential connectives to make up a first-order logic? What makes a second-level predicate or relation into a first-order logical quantifier? (analogously for higher-order quantifiers.)

To ask these questions is to investigate various criteria for logical "quantifierhood" with respect to their philosophical significance, formal results, and linguistic plausibility. At the two end points of the spectrum of possible criteria we find those that allow any second-level predicate and relation as a logical quantifier (this is possible within Frege's scheme because of the syntactic structure of second-level expressions) and those that allow only the universal and existential quantifiers as logical quantifiers. While the former amount to a trivialization of logic, the latter preclude extension altogether. For that reason, the area in-between is the most interesting for a critical investigator.

The formal scheme whose extension is considered in this book is Fregean-Russellian mathematical logic with Tarskian semantics. More specifically, the investigation concerns Tarski's model-theoretic semantics for first-order logic. With respect to Tarski's semantics one may wonder whether it makes sense at all to consider its extension to a logic with new quantifiers. To begin with, we can see that the structure of a first-order Tarskian model does allow for a definition of truth (via satisfaction) for a language richer than that of standard first-order logic. One line of reasoning pointing to the naturalness of extension is the following: If Tarskian model-theoretic semantics is philosophically correct, then a first-order model offers a faithful and precise mathematical representation of truth (satisfaction) conditions for a first-order extensional language. The formal correctness of this semantics ensures that no two distinct (i.e., logically nonequivalent) sets of sentences have mathematically indistinct classes of models. But ideally, a semantic theory would also be nonredundant, in the sense that no two distinct semantic structures would represent the same (i.e., logically equivalent) sets of sentences. Standard first-order logic does not measure up to this ideal, because it is unable to distinguish between nonisomorphic structures in general. That is, "elementary equivalence," equivalence as far as first-order theories go, does not coincide with equivalence up to isomorphism, a relation that distinguishes any two nonisomorphic structures.[6] This inadequacy can be "blamed" either on the excesses of the model-theoretic semantics or on the scantiness (expressive "poverty") of the standard first-order language. Accordingly, we can either make the semantic apparatus less distinctive or strengthen the expressive power of the standard language so that the model-theoretic semantics is put to full use. In any case, it is clear that Tarskian semantics can serve a richer language.

The study of extensions of logic has philosophical, mathematical, and linguistic aspects. Philosophically, my goal has been to find out what distinguishes logical from nonlogical terms, and, on this basis, determine the scope of (core) logic. Once the philosophical question has been decided, the next task is to delineate a complete system of first-order logic, in a sense analogous to that of the expressive completeness of various systems of truth-functional logic. In the early days of modern logic, truth-functionality was identified as the characteristic property of the "logical" sentential connectives, and this led to the semantics of truth tables and to the correlation of truth-functional connectives with Boolean functions from finite sequences of truth values to a truth value. That in turn enabled logicians to answer the question, What are all the truth-functional sentential connectives? and to determine the completeness (or incompleteness) of various sets of connectives. We cannot achieve the same level of effectiveness in the description of quantifiers. But we can try to characterize the logical quantifiers in a way that will reflect their structure (meaning), show how to "calculate" their value for any set of predicates (relations) in their domain, and describe the totality of quantifiers as the totality of functions of a certain kind. This task takes the form of construction, description, or redefinition, depending on whether the notion of a logical term that emerges out of the philosophical investigations has been realized by an existing formal system.

A further goal is a solid conceptual basis for the generalizations. In his *Introduction to Mathematical Philosophy* (1919) Russell says, "It is a principle, in all formal reasoning, to generalize to the utmost, since we thereby secure that a given process of deduction shall have more widely applicable results."[7] One of the lessons I have learned in the course of studying extensions of logic is that it is not always clear what the unifying idea behind a given generalization is or which generalization captures a given idea. In the case of generalized quantifiers, for example, it is not immediately clear what generalization expresses the idea of a *logical* quantifier. Indeed, the need to choose among alternative generalizations has been one of the driving forces behind my work.

Another angle from which I examine new forms of quantification is that of the ordering of quantifier prefixes: Why should quantifier prefixes be linearly ordered? Are partially-ordered quantifiers compatible with the principles of logical form? In his 1959 paper "Some Remarks on Infinitely Long Formulas," L. Henkin first introduced a new, nonlinear quantifier prefix (with standard quantifiers). Henkin interpreted his new quantifiers, *branching* or *partially-ordered quantifiers*, by means of Skolem functions.[8] An example of a branching quantification is

$$(1)\quad \begin{matrix}(\forall x)(\exists y)\\ (\forall z)(\exists w)\end{matrix} \!\!>\! \Phi(x, y, z, w),$$

which is interpreted, using Skolem functions, as

(2) $(\exists f^1)(\exists g^1)(\forall x)(\forall z)\Phi[x, f^1(x), z, g^1(z)]$.

However, attempts to extend Henkin's definition to generalized quantifiers came upon great difficulties. Only partial extensions were worked out, and it became clear that the concept of branching requires clarification. This is another case of a generalization in need of elucidation, and conceptual analysis of the branching structure is attempted in chapter 5.

The philosophical outlook underlying this work can be described as follows. Traditionally, logic was thought of as something to be discovered once and for all. Our thought, language, and reasoning may be improved in certain respects, but their logical kernel is fixed. Once the logical kernel is known, it is known for all times: we cannot change—improve or enrich—the logic of our language, reasoning, thought. On this view, questions about the logical structure of human language have definite answers, the same for every language. As the logical structure of human thinking is unraveled, it is encoded in a formal system, and the logical forms of this system are all the logical forms there are, the only logical forms. End of story.

This approach is in essence characteristic of many traditional philosophers, e.g., Kant in *Critique of Pure Reason* (1781/1787) and *Logic* (1800). The enterprise of logic, according to the *Critique*, consists in making an "inventory" of the "formal rules of all thought." These rules are simple, unequivocal, and clearly manifested. There is no questioning their content or their necessity for human thought. Because of the limited nature of its task, logic, according to Kant, "has not been able to advance a single step [since Aristotle], and is thus to all appearances a closed and completed body of doctrine."[9] That this view of logic is not accidental to

Kant's thought is, I think, evident from the use he makes of it in establishing the Table of Categories. The Table of Categories is based on the Table of the Logical Functions of the Understanding in Judgments, and the absolute certainty regarding the latter provides, according to Kant, an "unshakeable" basis for the former.

I, for one, do not share this view of logic. Even if there are "eternal" logical truths, I cannot see why there should be eternal conceptual (or linguistic) carriers of these truths, why the logical structure of human thought (language) should be "fixed once and for all." I believe that new logical structures can be constructed. Some of the innovations of modern logic appear to me more of the nature of invention than of discovery. Consider, for instance, Frege's construal of number statements. Was this a discovery of the form that, unbeknownst to us, we had always used to express number statements, or was it rather a proposal for a new form that allowed us to express number statements more fruitfully?

The intellectual challenge posed by man-made natural language is, to my mind, not only that of systematic description. As with mathematics or literature, the enterprise of language is first of all that of creating language, and this creative project is (in all three areas) unending. Even in contemporary philosophy of logic, most writers seem to disregard this aspect of language, approaching natural language as a "sacred" traditional institution. But does not the persistent, intensive engagement of these same philosophers with ever new alternative logics point beyond a search for new explanations to a search for new forms?

The view that there is no unique language of logic can also be based on a more conservative approach to human discourse. Defining the field of our investigation to be language as we currently use it, we can invoke the principle of multiformity of language, which is the linguistic counterpart of what H. T. Hodes called Frege's principle of the "polymorphous composition of thought."[10] Consider the following sentences:[11]

(3) There are exactly four moons of Jupiter.

(4) The number of moons of Jupiter = 4.

It is crucial for Frege, as Hodes emphasizes, that (3) and (4) express the same thought. The two sentences "differ in the way they display the composition of that thought, but according to Frege, one thought is not composed out of a unique set of atomic senses in a unique way."[12] Linguistically, this means that the sentence

(5) Jupiter has 4 moons,[13]

which can be paraphrased both by (3) and by (4), has both the logical form

(6) $(!4x)Ax$

and the logical form

(7) $(\iota x)Bx = 4$.

I think this principle is correct. Once we accept the multiformity of language, change in the "official" classification of logical terms is in principle licensed.

The logical positivists, unlike the traditional philosophers, made change in logic possible. Indeed, they made it too easy. Logic, on their view, is nothing more than a linguistic convention, and convention is something to be kept or replaced, at best on pragmatic grounds of efficiency (but also just on whim). I sympathize with Carnap when he says, "This [conventional] view leads to an unprejudiced investigation of the various forms of new logical systems which differ more or less from the customary form ..., and it encourages the construction of further new forms. The task is not to decide which of the different systems is 'the right logic' but to examine their formal properties and the possibilities for their interpretation and application in science."[14] Furthermore, I agree that accepting a new logic is adopting a new linguistic framework and that such "acceptance cannot be judged as being either true or false because it is not an assertion. It can only be judged as being more or less expedient, fruitful, conducive to the aim for which the language is intended."[15] What I cannot agree with is the insistence on the exclusively practical nature of the enterprise: "the introduction of the new ways of speaking does not need any theoretical justification ... to be sure, we have to face at this point an important question; but it is a practical, not a theoretical question."[16]

In my view, revision in logic, as in any field of knowledge, should face the "trial of reason" on both fronts, practice and theory. The investigations carried out in this essay concern the theoretical grounds for certain extensions of logic.

Generalized quantifiers have attracted the attention of linguists, and some of the most interesting and stimulating works on the subject come from that field. Quantifiers appear to be the closest formal counterparts of such natural-language determiners as "most," "few," "half," "as many as," etc. This linguistic perspective received its first elaborate and systematic treatment in Barwise and Cooper's 1981 paper "Generalized Quantifiers and Natural Language." Much current work is devoted to continuing Barwise and Cooper's enterprise.[17] The discovery of branching quantifiers in English is credited to J. Hintikka in "Quantifiers vs. Quantification Theory" (1973). Hintikka's paper aroused a heated discussion and steps towards a systematic linguistic analysis of branching quantifiers were taken by Barwise in "On Branching Quantifiers in English" (1979).

The work on generalized and branching quantifiers in linguistics, though answering high standards of formal rigor, has a strong empirical orientation. As a result, study of the "data" is given precedence over "pure" conceptual analysis. The task of formulating a cohesive empirical theory is particularly difficult in the case of branching quantifiers because evidence is so scarce. In fact, while the branching form appears to be grammatical, it is arguable whether it has, in actual languages, a clear semantic content. To me, this grammatical form appears to be "in search of a content." In any case, my own work emphasizes the conceptual aspect of the branching form. The direction of analysis is from philosophy to logic to natural language. This has the advantage, if the attempt is successful, that the theory is not piecemeal and the applications follow from a general conception. On the other hand, since empirical evidence is not given precedence, the proposals for linguistic applications are presented merely as theoretical hypotheses, and their empirical value is left for the linguist to judge.

My search for new logical forms is prompted by interests on several levels. For one thing, it is a way of asking the general philosophical questions: What is logic? Why should logic take the form of standard mathematical logic? For another, it is an attempt to understand more deeply the fundamental principles of modern logic. Mathematical logic, in particular first-order logic, has acquired a distinguished, paradigmatic place in contemporary analytic philosophy. This situation has naturally led to attempts to extend the range of its applicability, especially to various intensional contexts. It has also led to attacks on the basic principles of the standard system and to the consequent construction of alternative logics. Thus the philosophical scene abounds in modal, inductive, epistemic, deontic, and other extensions of "classical" first-order logic, as well as in intuitionistic, substitutional, free, and other rival logics. However, few in philosophy have suggested that the very principles underlying the "core" first-order logic might not be exhausted by the "standard" version. The present work ventures such a philosophical view, inspired by recent mathematical and linguistic developments. These have not yet received the attention they warrant in philosophical circles, and the opportunity they provide for a reexamination of fundamental principles underlying modern logic has largely passed unnoticed. The realization of

this opportunity motivates my work. Logic, I believe, is a vehicle of thought. This work is done with the hope of contributing to the understanding of its scope.

Chapter 2
The Initial Generalization

1 Mostowski and Frege

In the 1957 paper "On a Generalization of Quantifiers," A. Mostowski introduced linguistic operators of a new kind that, he said, "represent a natural generalization of the logical quantifiers."[1] Syntactically, Mostowski's quantifiers are formula building, variable binding operators similar to the existential and universal quantifiers of standard first-order logic. That is, if Φ is a formula, the operation of quantification by a Mostowskian quantifier Q and an individual variable x yields a more complex formula, $(Qx)\Phi$, in which x is bound by Q. Semantically, Mostowski's operators are functions that assign a truth value to any set of elements in the universe of a given model in such a way that the value assigned depends on the cardinalities of the set in question and its complement in the universe and on nothing else. Since the standard existential and universal quantifiers can also be defined in that manner, the new operators constitute a generalization of quantifiers in the semantic sense too. There thus exist, according to Mostowski, a great many operators on formulas with syntactic and semantic features similar to those of the standard quantifiers. These constitute a genuine extension of the logical quantifiers.

To understand Mostowski's generalization more deeply, I will begin with a short regression to Frege. Frege construed the existential and universal quantifiers as second-level quantitative properties that hold (or do not hold) of a first-level property in their range due to the size of its extension. This characterization of quantifiers is brought out most clearly in Frege's analysis of existence as a quantifier property in *The Foundations of Arithmetic* (1884): "Existence is a property of concepts."[2] "Affirmation of existence is in fact nothing but denial of the number nought."[3] "The proposition that there exists no rectangular equilateral rectilinear triangle

... state[s] a property of the concept 'rectangular equilateral rectilinear triangle'; it assigns to it the number nought."[4]

Within Frege's hierarchy of levels a (first-order) quantifier is a 1-place second-level predicate the argument place of which is to be filled by a 1-place first-level predicate (the argument place of which is in turn to be filled by a singular term). A sentence of the form $(\exists x)\Phi x$ is true if and only if (henceforth, "iff") the extension of the 1-place predicate (or propositional function) $\Phi\xi$ is of cardinality larger than 0. And a sentence of the form $(\forall x)\Phi x$ is true iff the extension of $\Phi\xi$ is the whole universe, or its counterextension has cardinality 0.[5]

This Fregean conception of the standard quantifiers underlies Mostowski's generalization. In Mostowski's model-theoretic terminology, the standard quantifiers are interpreted as functions on sets (universes of models) as follows:

(1) The universal quantifier is a function $\forall$ such that given a set A, $\forall(A)$ is itself a function $f: \boldsymbol{P}(A) \rightarrow \{T, F\}$, where $\boldsymbol{P}(A)$ is the power set of A and for any subset B of A,

$$f(B) = \begin{cases} T & \text{if } |A - B| = 0 \\ F & \text{otherwise.} \end{cases}$$

(2) The existential quantifier is a function $\exists$ such that given a set A, $\exists(A)$ is a function $g: \boldsymbol{P}(A) \rightarrow \{T, F\}$, where for any subset B of A,

$$g(B) = \begin{cases} T & \text{if } |B| > 0 \\ F & \text{otherwise.} \end{cases}$$

On the basis of this definition we can associate with each quantifier Q ($\forall$ or $\exists$) a function t^{Q} that tells us, given the size of a universe A, under what numerical conditions Q gives a subset B of A the value "true." Thus, the function t^{Q} may be defined as a function on cardinal numbers (sizes of universes) assigning to each cardinal number α another function t^{Q}_{α} that says how many objects are allowed to fall under a set B and its complement in a universe of size α in order for Q(B) to be "true." Since the "cardinality image" of each set in a universe of size α can be encoded by a pair of cardinal numbers $\langle \beta, \gamma \rangle$, where β represents the size of B and γ the size of its complement in the given universe, t^{Q}_{α} is defined as a function from all pairs of cardinal numbers β and γ, the sum of which is α, to $\{T, F\}$. So the universal quantifier function, $t^{\forall}$, is defined, for each α, by $t^{\forall}_{\alpha}$, which assigns to any given pair $\langle \beta, \gamma \rangle$ in its domain a value according to the rule

(3) $$t^{\forall}_{\alpha}(\beta, \gamma) = \begin{cases} T & \text{if } \gamma = 0 \\ F & \text{otherwise.} \end{cases}$$

The rule for the existential quantifier is

(4) $t^{\exists}_{\alpha}(\beta, \gamma) = \begin{cases} \text{T} & \text{if } \beta > 0 \\ \text{F} & \text{otherwise.} \end{cases}$

We can now define the standard quantifiers in terms of their t-functions as follows: Given a set A and a subset B of A,

(5) $\forall_A(B) = \begin{cases} \text{T} & \text{if } t^{\forall}_{|A|}(|B|, |A - B|) = T \\ \text{F} & \text{otherwise.} \end{cases}$

Similarly,

(6) $\exists_A(B) = \begin{cases} \text{T} & \text{if } t^{\exists}_{|A|}(|B|, |A - B|) = T \\ \text{F} & \text{otherwise.} \end{cases}$

However, $\forall$ and $\exists$ are not the only quantifiers that can be defined by cardinality functions like those above. Any function t that assigns to each cardinal number α a function t_α from pairs of cardinal numbers $\langle \beta, \gamma \rangle$ such that $\beta + \gamma = \alpha$ to $\{\text{T}, \text{F}\}$ defines a quantifier. Given a set A and a subset B of A, this quantifier is defined on A exactly as $\forall$ and $\exists$ are. For example, suppose that the cardinality function t^δ is defined, for any cardinal number α and pair $\langle \beta, \gamma \rangle$ such that $\beta + \gamma = \alpha$, by

(7) $t^{\delta}_{\alpha}(\beta, \gamma) = \begin{cases} \text{T} & \text{if } \beta = \delta \\ \text{F} & \text{otherwise.} \end{cases}$

Then t^δ determines the cardinal quantifier $(!\delta x)$: "for exactly δ elements x in the universe." Similar functions define the quantifiers "for at least δ elements x in the universe" and "for at most δ elements x in the universe." Cardinality statements in general, "δ things have property P," can thus be formalized as first-order quantifications

(8) $(!\delta x)Px$,

which assert that the extension of P has δ elements. In Frege's conceptual scheme, (8) would be a second-level statement that assigns a second-level numerical property to the extension of the first-level predicate P. But this is exactly Frege's own analysis of statements of number: "The content of a statement of number is an assertion about a concept. . . . If I say 'Venus has 0 moons' . . . what happens is that a property is assigned to the *concept* 'moon of Venus,' namely that of including nothing under it. If I say 'the King's carriage is drawn by four horses,' then I assign the number four to the concept 'horse that draws the King's carriage.'"[6] We see that Mostowski's generalization is indeed in the spirit of Frege.

Yet numerical quantifiers (finite and infinite) do not exhaust Mostowski's definition. Consider the function t defined (relative to a cardinal

number α and any pair $\langle \beta, \gamma \rangle$ such that $\beta + \gamma = \alpha$) as

(9) $t_{\alpha}(\beta, \gamma) = \begin{cases} \text{T} & \text{if } \beta > \gamma \\ \text{F} & \text{otherwise.} \end{cases}$

This function defines the quantifier (Mx), "most of the objects in the universe are such that . . ." (where we take "Most things are B" to mean "There are more Bs than non-Bs"). Consider also

(10) $t_{\alpha}(\beta, \gamma) = \begin{cases} \text{T} & \text{if } \beta \text{ is a finite even number} \\ \text{F} & \text{otherwise.} \end{cases}$

This function defines the quantifier (Ex): "an even number of objects in the universe are such that . . . ." Another quantifier is defined

(11) $t_{\alpha}(\beta, \gamma) = \begin{cases} \text{T} & \text{if } \beta = \alpha \\ \text{F} & \text{otherwise.} \end{cases}$

This is the *Chang* or *equicardinal quantifier*: "as many objects as there are elements in the universe are such that . . . ."[7] And so on.

Among the totality of cardinality functions t are functions that assign to different cardinals α different functions t_α. Such a "vacillating" function t might be defined for two distinct cardinal numbers α_1 and α_2 by

(12) $t_{\alpha_1}(\beta, \gamma) = \begin{cases} \text{T} & \text{if } \beta = m \\ \text{F} & \text{otherwise,} \end{cases}$

(13) $t_{\alpha_2}(\beta, \gamma) = \begin{cases} \text{T} & \text{if } \beta = n \\ \text{F} & \text{otherwise,} \end{cases}$

where $m \neq n$. The function t expresses cardinality properties of sets relative to the size of the universe: "m out of α_1, n out of α_2, ..." Some vacillating functions are reducible to "simple" functions like the ratio function "1/2," which is fixed for all universes. (Thus, $1/2 = 1$ out of 2, 2 out of 4, 3 out of 6, etc., where some conventional rule is given for universes with an odd number of elements.) Other vacillating functions represent irregular ratios ("2 out of 3, 3 out of 6, 19 out of 19, . . ."), and these are genuinely "manifold" cardinality functions.

According to Mostowski, any formula-binding operator defined by some cardinality function (simple or vacillating) as described above is a generalized quantifier.

2 A Criterion For Logical Quantifiers

Are Mostowski's quantifiers *logical* quantifiers? Are they *all* the logical quantifiers? From a Fregean point of view, standard first-order logic is a first-level system with one or two 1-place second-level predicates: the existential and/or universal quantifiers. Mostowski's logic is, from this point of view, a first-level system with an arbitrary number of 1-place second-level predicates of the same type as the standard quantifiers (i.e., 1-place second-level predicates of 1-place first-level predicates). However, not all second-level predicates of that type are logical quantifiers, according to Mostowski's definition. The predicate "*P* is a (first-level) attribute of Napoleon" is not. More generally, all noncardinality predicates are excluded from this category. The question naturally arises as to why the distinction between logical and nonlogical predicates should coincide with the distinction between cardinality properties and noncardinality properties. What does cardinality have to do with logicality?

Mostowski's answer is that there are two natural conditions on logical quantifiers:

CONDITION LQ1 "Quantifiers enable us to construct propositions from propositional functions."[8]

CONDITION LQ2 A logical quantifier "does not allow us to distinguish between different elements of [the universe]."[9]

The first requirement is clear. Syntactically, a quantifier is a formula-building expression that operates by binding a free variable in the formula to which it is attached and thus in finitely many applications generates a sentence, i.e., a closed formula.

The second, semantic requirement Mostowski interprets as follows: a 1-place first-level propositional function Φx satisfies a quantifier Q in a given model $\mathfrak{A}$ only if any 1-place first-level propositional function whose extension in $\mathfrak{A}$ can be obtained from that of Φx by some permutation of the universe satisfies it as well. More succinctly, logical quantifiers are invariant under permutations of the universe in a given model for the language.

It is interesting to note that (LQ2) is also suggested by Dummett in *Frege: Philosophy of Language* (1973):

> Let us call a *second-level condition* any condition which, for some domain of objects, is defined, as being satisfied or otherwise, by every predicate which is in turn defined over that domain of objects. Among such second-level conditions, we may call a *quantifier condition* any which is invariant under each permutation of the domain of objects: i.e. for any predicate '$F(\xi)$' and any permutation φ, it satisfies '$F(\xi)$' just in case it satisfies that predicate which applies to just those objects $\varphi(a)$, where '$F(\xi)$' is true of a. Then we allow as also being a logical constant any expression which ... allows us to express a quantifier condition which could not be expressed by means of ... [the universal and existential] quantifiers and the sentential operators alone.[10]

Now it is a metatheoretical fact about first-order models that given a model $\mathfrak{A}$ with a universe A and a 1-place second-level property $\mathscr{P}$, $\mathscr{P}$ satisfies (LQ2) with respect to the elements of A iff $\mathscr{P}$ is a cardinality property. And this explains why Mostowski identifies ***logical quantifiers*** with *cardinality quantifiers*. (A theorem establishing the one-to-one correspondence between quantifiers satisfying (LQ2) and cardinality quantifiers was proved by Mostowski. See the appendix.)

To sum up, syntactically, a quantifier is an operator binding a formula by means of an individual variable. Semantically, it is a function that assigns to every universe A an *A-quantifier* (or a *quantifier on A*), Q_A. Q_A is itself a function from subsets of A into $\{T, F\}$. We will call the cardinality function t associated with a given A-quantifier Q_A the *cardinality counterpart* of Q_A and symbolize it by t_A^Q (or sometimes simply by t). Quantifiers satisfying (LQ1) and (LQ2) are *Mostowskian quantifiers*. More precisely, a Mostowskian quantifier is a quantifier Q satisfying (LQ1) and such that for every set A, Q_A satisfies (LQ2), and if A_1, A_2 are sets of the same cardinality, then Q_{A_1} and Q_{A_2} have the same cardinality counterpart. For exact definitions, see the appendix.

It is worthwhile to note that Mostowski's system of generalized quantifiers *exhausts* the 1-place second-level predicates that satisfy (LQ2) only relative to the standard semantics for first-order logic. Disregarding the particular features of this semantics, we can say that any second-level predicate embodying *some measure* of sets and insensitive to the identity of their members satisfies this condition. Mostowski's quantifiers express measures of a particular kind, namely measures that have to do with the *cardinality* of sets, and as we have seen, these are all the second-level 1-place "measure predicates" satisfying (LQ2) relative to standard model theory. But these are not the only second-level measures conforming to (LQ2). Other quantifier measures of first-level extensions have been developed involving more elaborate model structures. Barwise and Cooper (1981) describe two such cases. The first is a quantifier Q, studied by Sgro (1977), where "$(Qx)\Phi x$" says that the extension of Φx contains a non-empty open set. This quantifier requires that models be enriched by some measure of distance (topology). The second has to do with measures of infinite sets: "Measures have been developed in which (a) and (b) make perfectly good sense.

(a) More than half the integers are not prime.
(b) More than half the real numbers between 0 and 1, expressed in decimal notation, do not begin with 7."[11]

Under the same category fall probability quantifiers, defined over nonstandard models in which probability values are assigned to extensions of predicates.[12] We will not take up quantifiers for nonstandard models here. We will also limit ourselves to finitistic logics. This is because the extension to infinitely long formulas is not necessary to investigate the generalized notion of a logical term, which is what interests us in this work.[13]

To return to Mostowski, the syntax of a first-order logic with (a finite set of) Mostowski's generalized quantifiers is the same as the syntax of standard first-order logic with two exceptions: (1) The language includes finitely many quantifier symbols, $Q_1, Q_2, \ldots, Q_n$ (among them possibly, but not necessarily, $\forall$ and $\exists$). (2) The rule for building well-formed quantified formulas is, If Φ is a well-formed formula, then $(Q_1x)\Phi$, $(Q_2x)\Phi, \ldots,$ $(Q_nx)\Phi$ are all well-formed formulas (for any individual variable x).

We can extend the Tarskian definition of satisfaction to cardinality quantifiers by replacing the entry for quantified formulas by the following. Let $\mathfrak{A}$ be a (standard) first-order model, and let A be the universe of $\mathfrak{A}$. Let g be an assignment of members of A to the variables of the language.

- If Φ is a formula and Q a quantifier symbol, $\mathfrak{A} \models (Qx)\Phi[g]$ iff for some α and β such that $\alpha + \beta = |A|$ and $t_A^Q(\alpha, \beta) = T$, there are exactly α elements $a \in A$ such that $\mathfrak{A} \models \Phi[g(x/a)]$ and exactly β elements $b \in A$ such that $\mathfrak{A} \models \sim\Phi[g(x/b)]$,

where "$\mathfrak{A} \models \Phi[g]$" is to be read, "$\Phi$ is satisfied in $\mathfrak{A}$ by g" and $g(x/a)$ is an assignment of members of A to the variables of the language that assigns a to x and otherwise is the same as g. Informally, the definition now says that $(Qx)\Phi x$ is true in $\mathfrak{A}$ iff the number of elements in A satisfying Φx and the number of elements in A not satisfying Φx are as t_A^Q allows. Note the following:

- The definition of satisfaction above is a schema that, for any given quantifier, instantiates differently. In the cases of $\forall$ and $\exists$ the schema instantiates in the standard way. In the case of the quantifier "most" the definition is, if Φ is a formula, then $(Mx)\Phi x$ is true in $\mathfrak{A}$ iff the number of Φx's in $\mathfrak{A}$ is larger than the number of non-Φx's in $\mathfrak{A}$. (Formally, $\mathfrak{A} \models (Mx)\Phi[g]$ iff for some α, β such that $\alpha + \beta = |A|$ and $\alpha > \beta$, there are exactly α elements a in A such that $\mathfrak{A} \models \Phi[g(x/a)]$ and exactly β elements b in A such that $\mathfrak{A} \models \sim\Phi[g(x/b)]$.)

- The definition of a model for first-order logic with Mostowskian generalized quantifiers is the same as that for standard first-order logic. A model for the extended logic does not contain any new "entities" not found in models for standard logic. The only difference is in the computation of truth values for quantified formulas, given a model and an assignment of objects in the universe to the variables of the language.

An important difference between Mostowski's system and standard first-order logic is that the former is not in general complete. Thus, for example, Mostowski proved that if the quantifiers of a generalized first-order logic include the existential or universal quantifier and at least one quantifier Q satisfying condition (*) below, then the logic is incomplete: not all logically true sentences of this system are provable, or, the set of sentences true in all models for the logic is not recursively enumerable.

CONDITION (*) The cardinality function t^Q associated with Q assigns to $\aleph_0$ a function $t^Q_{\aleph_0}$ such that both $\{n : t^Q_{\aleph_0}(n, \aleph_0) = F\}$ and $\{m : t^Q_{\aleph_0}(m, \aleph_0) = T\}$ are denumerable.[14]

An example of a quantifier satisfying condition (*) is Q, where, given a denumerable universe, "$(Qx)\Phi x$" means "The number of Φx's is even."

On the other hand some generalized logics are complete. For example, H. J. Keisler (1970) proved that the logic obtained from standard first-order logic with identity by adding the quantifier "there are uncountably many" and a modest set of axiom schemas is complete.

Referring to the incompleteness of first-order logic with generalized quantifiers in general, Mostowski says, "In spite of this negative result we believe that some at least of the generalized quantifiers deserve a closer study and some deserve even to be included into systematic expositions of symbolic logic. This belief is based on the conviction that the construction of formal calculi is not the unique and even not the most important goal of symbolic logic."[15]

One goal for which completeness appears to be immaterial is the characterization of the logical structure of natural language.

3 Generalized Quantifiers and Natural Language

In their seminal paper "Generalized Quantifiers and Natural Language" J. Barwise and R. Cooper examined Mostowski's theory from a linguistic perspective. Mostowski's logic, Barwise and Cooper observe, is superior to standard first-order logic in its account of natural-language quantification. "The quantifiers of standard first-order logic are inadequate for treating the quantified sentences of natural languages" in part because "there are sentences which simply cannot be symbolized in a logic which is restricted to ... ∀ and ∃."[16] Mostowski's method, on the other hand, allows us to encode the structure of such sentences as defy the standard analysis. Let me give a few examples:

(14) There are only a finite number of stars.

(15) No one's heart will beat an infinite number of times.[17]

(16) There is an even number of letters in the English alphabet.

(17) The number of rows in a (full) truth table is a power of 2.

(18) There are $2^{\aleph_0}$ reals between any two nonidentical integers.

The formal structure of (14) to (18) is analyzed in Mostowski's logic as follows:

(19) (Finite x) star x,

where for each model $\mathfrak{A}$ with universe A, $t_A^{\text{finite}}(\alpha, \beta) = \text{T}$ iff $\alpha < \aleph_0$;

(20) $\sim(\exists x)(\exists y)[y$ is a heart of x & (Inf z) z is a beat of $y]$,

where $t_A^{\text{inf}}(\alpha, \beta) = \text{T}$ iff $\alpha \geq \aleph_0$;

(21) (Even x) x is a letter in the English alphabet,

where t_A^{even} is defined by (10) above;

(22) $(\forall y)[y$ is a (full) truth table $\rightarrow$ (Power-of-2 x) x is a row in $y]$,

where $t_A^{\text{power-of-2}}(\alpha, \beta) = \text{T}$ iff α is a power of 2;

(23) $(\forall x)(\forall y)[x$ is an integer & y is an integer & $x \neq y \rightarrow (2^{\aleph_0}z)(z$ is a real number & z is between x and $y)]$,

where $t_A^{2^{\aleph_0}}(\alpha, \beta) = \text{T}$ iff $\alpha = 2^{\aleph_0}$.

What about (24) to (29)?

(24) More than one third of the population of the world suffers from hunger.

(25) 94% of all Americans believe in God.

(26) Some recipients of a Nobel prize are known to most people in the world.

(27) Most people are not hostile to most people.

(28) As many Israelis are liberals as are not.

(29) $\aleph_0$ natural numbers are prime, and the same number are not prime.

Do these resist a first-order symbolization? No, say Barwise and Cooper. By inserting nonlogical, "domain-fixing" axioms of the form $(\forall x)\Phi x$, by introducing many-sorted variables, or by limiting consideration to particular models, one can use Mostowski's quantifiers to analyze natural-language sentences like (24) to (29). Thus we might formalize (24) to (29) as follows:

(30) (More-than-1/3 x) x is suffering from hunger,

where $t_A^{\text{more-than-1/3}}(\alpha, \beta) = \text{T}$ iff $\alpha > 1/3(\alpha + \beta)$, both α and β are finite, and the range of x is the set of all people in the world (at the present, say).

(31) (94% x) x believes in God,

where $t_A^{94\%}(\alpha, \beta) = \text{T}$ iff $\alpha = 94\% \ (\alpha + \beta)$, α, β are finite, and the range of x is the set of all American people (at the present).

(32) $(\exists x)[x$ is a recipient of a Nobel prize & (Most y) x is known to $y]$

Here t_A^{most} is defined by (9) above. The range of x is as in (30).

(33) a. $\sim$(Most x)(Most y) x is hostile to y

b. (Most x) $\sim$ (Most y) x is hostile to y

c. (Most x)(Most y) $\sim$ (x is hostile to y)

whether the analysis is (a), (b), or (c) depends on how you read the negation in (27). Both t_A^{most} and the range of x are as above.[18]

(34) (As-many-as-not x) x is a liberal,

where $t_A^{\text{as-many-as-not}}(\alpha, \beta) = \text{T}$ iff $\alpha \geq \beta$, and the range of x is the set of Israelis.

(35) $(\aleph_0/\aleph_0\ x)$ x is prime,

where $t_A^{\aleph_0/\aleph_0}(\alpha, \beta) = \text{T}$ iff $\alpha = \beta = \aleph_0$, and the range of x is the set of natural numbers.

Clearly, the natural-language "most," "almost all," "few," "a few," "many," etc. can be construed as Mostowskian quantifiers only to the extent that they can be given absolute cardinality values (or ranges of values). Under such a construal, we read "most" as "(cardinalitywise) more than a half," just as in standard logic we read "some" as "at least one."

What are the limitations of Mostowski's system from the point of view of the logical structure of natural language? Consider the following sentences:

(36) Most of John's arrows hit the target.[19]

(37) 60% of the female students in my class are A-students.

(38) The majority of children who do not communicate with anyone during the first two years of their lives are autistic.

(39) Most of the students in most colleges are not exempt from tuition fees.

Delimiting the range of the bound variables, which enabled us to analyze (24) to (29), is inadequate for the formalization of (36) to (39). Restricted or sorted domains are useful up to the point where they become utterly artificial, as they would if they were used to analyze (36) to (39) with Mostowski's quantifiers. Mostowski's system is rich enough to analyze sentences of the form

(40) Such and such a quantity of all the objects that there are, are B,

but in general is inadequate for the analysis of sentences of the form

(41) Such and such a quantity of all As are Bs.

This verdict was reached both by Barwise and Cooper and by N. Rescher in "Plurality-Quantification" (1962) and elsewhere.[20] Clearly, statements of type (41) cannot be symbolized as

(42) $(Qx)(Ax \rightarrow Bx)$,

as can be seen by the following counterexample: Suppose that only one third of the things that satisfy Ax in a given model $\mathfrak{A}$ satisfy Bx in $\mathfrak{A}$. Suppose also that most of the things in the universe of $\mathfrak{A}$ do not satisfy Ax. Then $(Mx)(Ax \rightarrow Bx)$ will come out true in $\mathfrak{A}$ (most things in $\mathfrak{A}$ satisfy "$Ax \rightarrow Bx$" by falsifying the antecedent), although it is plainly false that most of the As in $\mathfrak{A}$ are Bs.

In general, a statement of the form (41) cannot be formalized by a formula of the form

(43) $(Qx)\Phi x$.

(A theorem to the effect that "more than half of the As" cannot be defined in terms of "more than half of all things that there are" using the apparatus of standard first-order logic was proved by Barwise and Cooper for finite universes and by D. Kaplan for the infinite case.[21]) Rescher concludes,

> Textbooks often charge that traditional logic is "inadequate" because it cannot accommodate patently valid arguments like (1) [All A's are B's $\vdash$ All parts of A's are parts of B's]. But this holds equally true of modern quantificational logic itself, which cannot accommodate (2) [Most things are A's; Most things are B's $\vdash$ Some A's are B's] until supplemented by something like our plurality-quantification [Mostowski's "most"]. And even such expanded machinery cannot accommodate (3) [Most C's are A's; Most C's are B's $\vdash$ Some A's are B's]. Powerful tool though it is, quantificational logic is unequal to certain childishly simple valid arguments.[22]

Barwise and Cooper's strategy in the face of the alleged inexpediency of quantificational logic is to give up logical quantifiers altogether. The idea underlying this move seems to be the following: There is no absolute meaning to such expressions as "more than half." The quantities involved in "more than half the natural numbers between 0 and 10" are different from those involved in "more than half the natural numbers between 0 and 100." Hence "more than half" cannot be interpreted independently of the interpretation of the set expression attached to it. Thus in the schema

(44) More than half the As are Bs,

"'more than half' is not acting like a quantifier, but like a determiner. It combines with a set expression to produce a quantifier."[23] "*Quantifiers correspond to noun-phrases, not to determiners.*"[24] The quantifier in (44) is the whole noun phrase "more than half the As," and (44) is rendered

(45) (More-than-half-A x)Bx.[25]

In this way the indeterminacy inherent in determiners is resolved by the set expressions attached to them, and the difficulty indicated above disappears: "more than half the natural numbers between 0 and 10" and "more than half the natural numbers between 0 and 100" are two distinct quantifiers, each with its own meaning. And in general, quantifiers are pairs, $\langle D, S \rangle$, of a determiner D and a set expression S. If S, S' denote different sets, DS and DS' are different quantifiers.

What about "every" and "some"? According to Barwise and Cooper, the situation is indeed different in the case of "every." The schema

(46) Every A is B

can be expressed in terms of the quantifier "every" independently of the interpretation of A:

(47) (Every x)$(Ax \rightarrow Bx)$.

However, they say, the syntactic dissimilarity of (46) and (47) indicates that even in this case the "true" quantifier is "every A." "Every" is but a determiner, although, unlike "more than half," it is a logical determiner. Sentence (46), then, is to be symbolized not as (47) but rather as

(48) (Every-A x)Bx.

4 Nonlogical Quantifiers

As a theory of quantification, Barwise and Cooper's theory is evidently very bloated. "Every man," "every woman," "every child," "every son of mine," etc. are all different quantifiers. So are "most men," "most women," "most children," and so on. Two questions present themselves: Is such an excessive theory of quantifiers necessary to account for the diverse patterns of quantification in natural-language discourse? Is what this theory explains quantification?

Barwise and Cooper address the following questions:

- What is the role of quantifiers and how are they interpreted in a model? According to Barwise and Cooper, we use quantifiers to attribute properties to sets. "$\exists x\varphi(x)$" asserts that the set of things satisfying $\varphi(x)$ is not empty. "$\forall x\varphi(x)$" says that the set of φs contains all the objects in the universe of discourse. "Finite $x\varphi(x)$" states that this set is finite. And so on. Model-theoretically, a quantifier partitions the "family" of all subsets of the universe of a given model into those that satisfy it and those that do not. When combined with the former, it yields the value T, when combined with the latter, the value F. Thus a quantifier can be identified with the family of all sets to which it gives the value T. (Note that according to this account *all* properties of sets are quantifier properties.)
- What is the syntactic category of the natural-language expressions that function as quantifiers? Barwise and Cooper observe that noun phrases in general behave like quantifiers. Given a noun phrase, some verb phrases will combine with it to produce true sentences, and others to produce false sentences. Semantically, this means that each noun phrase divides the family of verb-phrase denotations in a given model into two groups: those that satisfy it and those that do not. Therefore, Barwise and Cooper conclude, "the noun phrases of a language are all and only the quantifiers over the domain of discourse."[26] To make their treatment of noun phrases uniform, Barwise and Cooper have to show that proper names can also be treated as quantifiers. But this is not difficult to show. We can treat a proper name like "Harry" as partitioning all the sets in the universe into those that contain Harry and those that do not. Thus "Harry" can be semantically identified with the family of all sets that include Harry as a member. "In our logic," Barwise and Cooper say, "(a) may be translated as (b), or rather, something like (b) in structure.

(a) Harry knew he had a cold.

(b) Harry $\hat{x}[x$ knew x had a cold]."[27]

In sum, "*Proper names and other noun-phrases are natural language quantifiers.*"[28]

The linguistic logic developed by Barwise and Cooper and based on the above principles differs from standard first-order logic and its Mostowskian

extension in several substantive ways. We can outline its main features as follows:

Syntactically, the logic excludes logical quantifiers altogether. Instead, it includes logical and nonlogical determiner symbols (the logical determiners include "some," "every," "no," "both," "1" (in the sense of "at least 1"), "2," "3," ..., "!1" ("exactly one"), "!2," ..., "the 1," ...; the nonlogical determiners include "most," "many," "few," "a few," ...). Quantifiers are nonlogical complex terms representing noun phrases in general: "John," "Jerusalem," "most people," "five boys," etc. Formally, quantifiers are defined terms of the form $D(\eta)$, where D is a determiner and η is a (first-level) predicate with a marked argument place called a "set term." A quantified formula is of the form Q(η), Q being a quantifier and η a set term. The language also includes the distinguished 1-place first-level predicate (set term) "thing."

Semantically, a model $\mathfrak{E}$ for the logic provides, in addition to the standard universe of objects, interpretations $\mathfrak{I}$ for the truth-functional connectives, "thing," and the logical as well as nonlogical determiners. "Thing" is interpreted as the universe of the model, E; each (logical or nonlogical) determiner is interpreted as a function that assigns to every set in the model a family of sets in the model. $\mathfrak{I}(!n) = \mathbf{!n} : \boldsymbol{P}(E) \to \boldsymbol{P}(\boldsymbol{P}(E))$ such that for each $A \subseteq E$, $\mathbf{!n}(A) = \{X \subseteq E : |A \cap X| = n\}$; $\mathfrak{I}(\text{Most}) = \mathbf{most} : \boldsymbol{P}(E) \to \boldsymbol{P}(\boldsymbol{P}(E))$ such that for each $A \subseteq E$, $\mathbf{most}(A) = \{X \subseteq E : |A \cap X| > |A - X|\}$; etc. The truth-functional connectives are interpreted in the usual way (although Barwise and Cooper favor a trivalent logic to allow for determiners denoting partial functions). Quantifiers (nonlogical terms) —$D(\eta)$ for some determiner D and set term η—are interpreted in each model as the family of sets assigned in this model by the denotation of D to the denotation of η. For example, $\mathfrak{I}(\mathbf{!n(man)}) = \{X \subseteq E : |\{x : x \text{ is a man}\} \cap X| = n\}$ and $\mathfrak{I}(\mathbf{John}) = \{X \subseteq E : \text{John} \in X\}$. If Φ and Ψ are 1-place predicate symbols (set terms), "$(D\Phi)[\Psi]$" is true in $\mathfrak{E}$ iff the denotation of Ψ in $\mathfrak{E}$ is a member of the family of sets assigned to $D\Phi$ in $\mathfrak{E}$.

Barwise and Cooper posit a universal semantic constraint on natural-language determiners: "It is a universal semantic feature of determiners that they assign to any set A a quantifier (i.e. family of sets) that *lives on* A," where Q *lives on* A iff for any set X, $X \in \text{Q}$ iff $A \cap X \in \text{Q}$.[29] The following equivalences illustrate this notion:

Many men run $\leftrightarrow$ Many men are men who run

Few women sneeze $\leftrightarrow$ Few women are women who sneeze

John loves Mary $\leftrightarrow$ John is John and loves Mary [30]

"The quantifiers represented by the subjects of the sentences," Barwise and Cooper explain, "live on the set of men, women and the singleton set containing John, respectively."[31] And they conclude, "When we turn to non-logical determiners, [the living on constraint] is the only condition we impose as part of the *logic*."[32] This condition on determiners is ipso facto a condition on quantifiers.

Is what Barwise and Cooper's theory explains *quantification*? To resolve this issue, let us consider several intermediate questions: In what sense is Barwise and Cooper's logic a first-order system, given that quantifiers are nonlogical second-level predicates? Quantifiers, in Barwise and Cooper's system, are 1-place predicates. Is being 1-placed an essential property of quantifiers? If so, why? How does Barwise and Cooper's criterion for "quantifierhood" compare with Mostowski's? Does their theory account for all natural-language quantifiers intuitively satisfying Mostowski's principles? Does it account only for such quantifiers?

Obviously, Barwise and Cooper's requirements on quantifiers are altogether different from those of Mostowski (and Dummett). In particular, Barwise and Cooper's quantifiers do not satisfy the semantic condition (LQ2). These quantifiers *do* in general distinguish elements in the universe of a model for their system. Consider the following two pairs of sentences:

(49) a. Einstein $\hat{x}[x$ is among the ten greatest physicists of all time$]$

b. Einstein $\hat{x}[x$ is among the ten greatest novelists of all time$]$

(50) a. Most (natural numbers between 1 and 10) $\hat{x}\ [x < 7]$

b. Most (natural numbers between 1 and 10) $\hat{x}\ [9 < x < 17]$.

Although the extension of "x is among the ten greatest physicists of all time" can be obtained from that of "x is among the ten greatest novelists of all time" by a permutation of the universe of discourse, the quantifier "Einstein" assigns the two sets different truth values. Similarly, "Most natural numbers between 1 and 10" assigns different truth values to the extensions of "$x < 7$" and "$9 < x < 17$" in spite of the fact that the one extension can be obtained from the other by some permutation of the (intended) universe.

Moreover, not all quantity properties, properties that satisfy Mostowski's criterion, are quantifiers (or constituents of quantifiers, i.e., determiners) on Barwise and Cooper's view. Thus the requirement that quantifiers "live on" the sets in their domain excludes some linguistic constructions that we would expect to be analyzed by means of cardinality quantifiers:

(51) Mostly women have been elected to Congress.

(52) Only human beings have brains.

(53) Not only men are allowed in the club.

Sentence (51) cannot be formalized as

(54) Mostly (women) $\hat{x}\ [x$ has been elected to Congress$]$,

since (54), unlike (51), comes out true in the intended model: the requirement of "living on" implies that for any set X, "Mostly (women)" is true of X if it is true of $\{x : x \text{ is a woman}\} \cap X$. But clearly, "Mostly women are women elected to Congress" is true. Similar problems occur if we want to formalize (52) as

(55) Only (human beings) $\hat{x}\ [x$ has a brain$]$,

or (53) as

(56) Not only (men) $\hat{x}\ [x$ is allowed in the club$]$.

The requirement of "living on" determines that (55) is true and (56) is false, while (52) is clearly false, and (53) is sometimes true. Barwise and Cooper can get over the difficulty by rephrasing (51) to (53) with determiners other than "mostly," "only," and "not only," respectively, but only at the expense of giving up the desideratum of transparent analysis of linguistic structures.

The requirement that only *noun phrases* be construed as quantifiers also blocks "natural" candidates for natural-language quantifiers. Consider the following sentences involving quantitative comparison of extensions, which, by Dummett's and Mostowski's criterion, have a good claim to being analyzed as statements of quantification:

(57) There are fewer men than women.

(58) More people die of heart disease than die of cancer.

(59) They are outnumbered by us.

(60) The same percentage of boys and girls who took the test received a perfect score.

Clearly, the operation of quantification in (57) to (60) is not carried out by noun phrases.

To sum up, Barwise and Cooper's theory is clearly not based on Mostowski's ideas of the nature of quantifiers. In particular, Mostowski's semantic condition, (LQ2), is violated by Barwise and Cooper. Their theory also does not offer an alternative principle to (LQ2) of the same general import as Mostowski's. In my opinion, Barwise and Cooper's analysis explains some linguistic regularities, but what it explains is not the structure of *quantifiers*. Their account is at once too particular to explain the notion of quantification in all its generality—witness (51) to (53) and (57) to (60)—and too general to focus on the unique features of quantifiers—see (49) and (50).[33]

5 Logical Quantifiers

Can we increase the expressive power of Mostowski's logical system so that it is no longer subject to Barwise and Cooper's criticism, without betraying its underlying principles?

I think the "inadequacy" of Mostowski's system can be analyzed along lines different from those taken by Barwise and Cooper. The problem is neither with the "logicality" of Mostowski's quantifiers nor with his criterion for second-level predicates expressing quantifier properties (a criterion shared, as we have seen, by Dummett). The problem is that Mostowski explicitly considered only *1-place* second-level predicates as candidates for quantifiers. Progress in logic was made after the indispensability of *relations* was acknowledged. Frege's revolution was in part in recognizing relations for what they are: *irreducibly many-place predicates*. Mostowski's requirements on quantifiers—that they turn propositional functions into propositions and that they do not distinguish elements in a model for the language—contain nothing to exclude many-place second-level predicates from being first-order quantifiers. On the contrary, the failure of Mostowski's theory to display the quantificational structure of sentences such as (36) to (39) is testimony only to the "incompleteness" of that theory. Mostowski's theory of cardinality quantifiers includes all the "predicative" quantifiers that express cardinality measures, but none of the "relational" quantifiers that express such measures. And there is no reason to believe that Mostowski would have rejected many-place quantifiers.

With this observation the solution to Barwise and Cooper's problem becomes very simple: Both "most" in "most things are *A*" and "most" in "most *A*s are *B*s" are quantifiers, although, as was proved by Barwise and Cooper, the second is not reducible to the first. The first is a 1-place quantifier, M^1, i.e., a property of first-level properties (or a 1-place function from first-level properties to truth values). It appears in formulas of the form

$$(M^1 x)\Phi x,$$

and for any given model $\mathfrak{A}$ with universe A it is defined by the funtion $t_A^{M^1}$ as in (9) above. That is, for all pairs of cardinals α, β whose sum is $|A|$,

$$t_A^{M^1}(\alpha, \beta) = \mathrm{T} \text{ iff } \alpha > \beta.$$

The second "most" is a 2-place quantifier, M^2, i.e., a 2-place *relation* between first-level properties (or a 2-place function from pairs of first-level properties to truth values). It appears in formulas of the form

$$(M^2 x)(\Phi x, \Psi x),$$

and for any given model $\mathfrak{A}$ with universe A it is definable by a function $t_A^{M^2}$, which I will presently characterize.

What kind of relation is M^2? It is a relation that in any given universe A holds between two subsets B, B' of A (in that order) iff $|B \cap B'| > |B - B'|$. M_A^2 is defined by the function t from quadruples of cardinals α, β, γ, δ such that $\alpha + \beta + \gamma + \delta = |A|$ to {T, F} as follows:

$$t_A^{M^2}(\alpha, \beta, \gamma, \delta) = \mathrm{T} \text{ iff } \alpha > \beta,$$

where, intuitively, $\alpha = |B \cap B'|$, $\beta = |B - B'|$, $\gamma = |B' - B|$, and $\delta = |A - (B \cup B')|$. Using a Venn diagram, we can form a visual image of the relation between the cardinal numbers α, β, γ, δ and parts of the universe A as in figure 2.1.

It is easy to see that whereas M^2 is not definable in terms of M^1, M^1 *is* definable in terms of M^2. Thus,

$$(M^1 x)\Phi x \text{ iff } (M^2 x)(x = x, \Phi x).$$

Our analysis provides a rationale for extending Mostowski's original system in a way that was first proposed by Per Lindström in "First order Predicate Logic with Generalized Quantifiers" (1966). This extension has been widely adopted by logicians and mathematicians, including Barwise in his purely logical writings. (Barwise later also expressed misgivings about treating proper names as quantifiers in natural-language analysis.[34]) Lind-

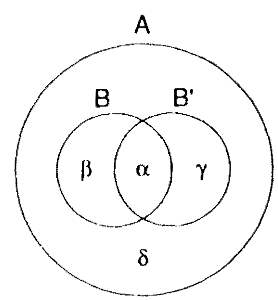

Figure 2.1

ström did not discuss the reasons underlying his extension of Mostowski's system, but as we have just seen, philosophico-linguistic considerations support his approach. In accordance with Lindström's proposal we add to Mostowski's original quantifiers all second-level 2-place relations of first-level 1-place predicates satisfying (LQ2). The one-to-one correlation between Mostowski's quantifiers and cardinality functions is preserved under the new extension. (See the appendix.)

The syntax and the semantics of first-order logic with 1- and 2-place generalized quantifiers is a natural extension of the syntax and the semantics of section 2 above. Again, a model for a language of this logic is the same as a model for a language of standard first-order logic: the "enrichment" is expressed by the rules for computing the truth values of formulas in a model (relative to an assignment of elements in the universe to the individual variables of the language).

I will now present a formal description of the extended logic.

First-Order Logic with 1- and 2-Place Generalized Quantifiers

Syntax

Logical symbols In addition to the logical symbols of standard first-order logic (but with the possible omission of $\forall$ and/or $\exists$) the language includes 1- and 2-place quantifier symbols: $Q^1_1, \ldots, Q^1_m$ and $Q^2_1, \ldots, Q^2_n$ for some positive integers m and n. (If $\forall$ and/or $\exists$ belong in the language, they fall under the category of 1-place quantifiers, and we add to them the superscript "1.")

Punctuation The usual punctuation symbols for first-order logic plus the symbol",".

Nonlogical symbols The same as in standard first-order logic.

Terms The same as in standard first-order logic.

Formulas The same definition as for standard first-order logic, but the definition of quantified formulas is replaced by the following:

(I) If Φ is a formula and Q^1 is a 1-place quantifier symbol, then $(Q^1x)\Phi$ is a formula.

(II) If Φ, Ψ are formulas and Q^2 is a 2-place quantifier symbol, then $(Q^2x)(\Phi, \Psi)$ is a formula.

Semantics

The semantics is the same as that for standard first-order logic, but the definition of satisfaction of quantified formulas in a model $\mathfrak{A}$ with a

universe A relative to an assignment g for the variables of the language is changed to the following:

(A) If Φ is a formula and Q^1 is a 1-place quantifier symbol,
$\mathfrak{A} \models (Q^1x)\Phi[g]$ iff for some cardinal numbers α and β such that $\alpha + \beta = |A|$ and $t_A^{Q^1}(\alpha, \beta) = T$, there are exactly α elements $a \in A$ such that $\mathfrak{A} \models \Phi[g(x/a)]$ and exactly β elements $b \in A$ such that $\mathfrak{A} \models {\sim}\Phi[g(x/b)]$.

(B) If Φ, Ψ are formulas and Q^2 is a 2-place quantifier symbol,
$\mathfrak{A} \models (Q^2x)(\Phi, \Psi)[g]$ iff for some cardinal numbers $\alpha, \beta, \gamma, \delta$ such that $\alpha + \beta + \gamma + \delta = |A|$ and $t_A^{Q^2}(\alpha, \beta, \gamma, \delta) = T$,
- there are exactly α elements $a \in A$ such that $\mathfrak{A} \models (\Phi \,\&\, \Psi)[g(x/a)]$,
- there are exactly β elements $b \in A$ such that $\mathfrak{A} \models (\Phi \,\&\, {\sim}\Psi)[g(x/b)]$,
- there are exactly γ elements $c \in A$ such that $\mathfrak{A} \models ({\sim}\Phi \,\&\, \Psi)[g(x/c)]$,
- there are exactly δ elements $d \in A$ such that $\mathfrak{A} \models ({\sim}\Phi \,\&\, {\sim}\Psi)[g(x/d)]$.

Applications

Within the system defined above we can easily analyze the logical structure of sentences that eluded Mostowski's original devices, like (36) to (39) above. We choose a language that includes, in addition to $\exists^1$ and M^2, the 2-place quantifier "60%," defined by $t(\alpha, \beta, \gamma, \delta) = T$ iff $\alpha + \beta < \aleph_0$ and $\alpha = 60\%(\alpha + \beta)$. We then symbolize (36) to (39) as given below:

(61) (M^2x)(x is an arrow of John's, x hits the target)

(62) $(60\%^2x)$(x is a female student in my class, x is an A student)

(63) (M^2x)[x is a child & $\sim(\exists^1y)$(x communicates with y during the first two years of x's life), x is autistic]

(64) (M^2x)[x is a college, (M^2y)(y is a student at x, $\sim$(y is exempt from tuition fees at x))]

By adding to the language the 2-place quantifiers

S^2, defined by $t(\alpha, \beta, \gamma, \delta) = T$ iff $\alpha > \gamma$,
O^2, defined by $t(\alpha, \beta, \gamma, \delta) = T$ iff $\gamma = 0$,
N^2, defined by $t(\alpha, \beta, \gamma, \delta) = T$ iff $\gamma \neq 0$,
F^2, defined by $t(\alpha, \beta, \gamma, \delta) = T$ iff $\beta < \gamma$, and
R^2, defined by $t(\alpha, \beta, \gamma, \delta) = T$ iff $\beta > \gamma$,

we can encode the logical structure of sentences (51) to (53) and (57) to (59), which were problematic for Barwise and Cooper:

(65) (S^2x)(x is a woman, x has been elected to Congress)

(66) $(O^2x)(x$ is a human being, x has a brain)

(67) a. $\sim(O^2x)(x$ is a man, x is allowed in the club)

b. $(N^2x)(x$ is a man, x is allowed in the club)

(68) $(F^2x)(x$ is a man, x is a woman)

(69) $(R^2x)(x$ is a person & x dies of heart disease, x is a person & x dies of cancer)

(70) $(F^2x)(x$ is one of them, x is one of us)

Within the new system we can also represent the logical structure of (50.a) and (50.b) without violating (LQ2):

(71) $(M^2x)(1 < x < 10, x < 7)$,

(72) $(M^2x)(1 < x < 10, 9 < x < 17)$.

We have seen an example of nested 2-place generalized quantifiers in (64). A more concise example is

(73) Most men love most women,

symbolized as

(74) $(M^2x)[x$ is a man, $(M^2y)(y$ is a woman, x loves $y)]$.

To formalize (60), we need to include 3-place quantifiers in the system. The reason is that (60) involves comparison between (subsets of) three sets: the set of all boys who took the test, the set of all girls who took the test, and the set of all those who received a perfect score in the test. We have not defined 3-place quantifiers for our formal system, but it is easy to see how this would be done.

A 3-place quantifier is a function that assigns to each universe A a 3-place quantifier on A, Q^3_A. Q^3_A is defined by a cardinality function t_A that takes into account all the "atoms" of Boolean combinations (intersections, unions, and complements) of triples, $\langle B, C, D\rangle$, of subsets of A. Since there are eight such atoms: $B \cap C \cap D$, $(B \cap C) - D$, $(C \cap D) - B$, $(D \cap B) - C$, $B - (C \cup D)$, $C - (B \cup D)$, $D - (B \cup C)$, and $A - (B \cup C \cup D)$, $t_A^{Q^3}$ is a function from 8-tuples, $\langle \alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta, \theta\rangle$, of cardinal numbers whose sum is $|A|$ to $\{T, F\}$. We need to decide on the order in which $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta, \theta$ represent (sizewise) the atoms generated by B, C, and D in A. I use a Venn diagram to fix a correlation (figure 2.2).

We can now formalize (60) as

(75) $(S^3x)(x$ is a boy who took the test, x is a girl who took the test, x received a perfect score in the test),

where S^3 is defined by a function t such that when $|A|$ is finite,

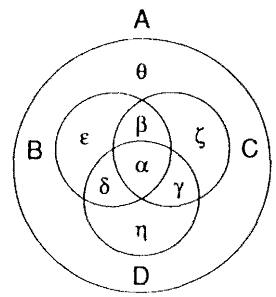

Figure 2.2

$$t(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta, \theta) = \mathrm{T} \text{ iff } \frac{\alpha + \delta}{\alpha + \beta + \delta + \varepsilon} = \frac{\alpha + \gamma}{\alpha + \beta + \gamma + \zeta}.$$

The present theory enables us to explain the syntactic differences between logical and linguistic quantifications, e.g., the linguistic

Every A is B

and the logical

$(\text{Every } x)(Ax \rightarrow Bx)$,

by conjecturing that natural language frequently employs quantifiers that, from a purely logical perspective, are redundant. Thus, whereas $\forall^2$ (defined by: $t(\alpha, \beta, \gamma, \delta) = \mathrm{T}$ iff $\beta = 0$) is logically superfluous, it may very well be the quantifier used in natural language in sentences of the form (46). The logical form of (46) is, on this conjecture,

(76) $(\text{Every}^2\ x)(Ax, Bx)$.

Likewise, we obtain a structure similar to that of

(77) Fewer men than women live to be 80

by employing a 3-place quantifier, F^3, rather than the 2-place quantifier F^2. Sentence (77) is then rendered

(78) $(F^3x)(x$ is a man, x is a woman, x lives to be 80)

instead of the logically simpler

(79) $(F^2x)(x$ is a man & x lives to be 80, x is a woman & x lives to be 80).

F^3 is defined, $t(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta, \theta) = \mathrm{T}$ iff $\delta < \gamma$. Logic with redundant many-place quantifiers can be seen as a bridge between the logical and the grammatical analysis of language.

The linguistic merits of the *logical*, as opposed to *nonlogical*, quantifier approach have been noted by linguists working in Logical Form (LF) research. An anonymous referee for this book indicates that the category of *logical quantifiers* (without the "living on" constraint) is significant not only semantically but also syntactically. Extensive research on LF has shown that "there are systematic syntactic differences between NPs depending upon whether they are logical or non-logical terms." Thus transparency between syntax and semantics favors logical, as opposed to non-logical, quantifiers. Among linguists whose work exemplifies the logical-quantifier approach are Higginbotham and May (1981) and May (1989, 1990). In addition, May (1991) cites reasons pertaining to language learnability for identifying quantifiers with cardinality operators and categorizing all quantifiers as logical terms. May writes,

> In distinguishing the logical elements in the way that we have, we are making cleavage between logical items whose meanings are formally, and presumably exhaustively, determined by UG [Universal Grammar]—the logical terms—and those whose meanings are underdetermined by UG—the non-logical, or content, words. This makes sense, for to specify the meaning of quantifiers, all that is needed, formally, is pure arithmetic calculation on cardinalities, and there is no reason to think that such mathematical properties are not universal. For other expressions, learning their lexical meanings is determined causally, and will be effected by experience, perception, knowledge, common-sense, etc. But none of these factors is relevant to the meaning of quantifiers. The child has to learn the content of the lexical entries for the non-logical terms, but this is not necessary for the entries for the logical terms, for they are given innately.[35]

The considerations adduced by May open a way to empirically ground not only the notion of quantifier developed so far but also the philosophical demarcation of logic in general as presented in this book.

A few words about the limitations of Mostowskian quantifiers. Some predicates of natural language are such that a proper representation of their extension is not possible in standard first-order model theory. Quantifier expressions do attach to such predicates, however. Here are two examples:

(80) Most of the water in the lake has evaporated.

(81) More arms than we have are needed to win this war.

"Water in the lake" and "arms needed to win this war" do not sort the objects in a universe into those that fall, and those that do not fall under them. Hence the present theory, which does not change the standard structure of first-order models, cannot account for their logical form.

In addition to predicates that defy first-order symbolization, we also find in natural language a use of quantifiers that exceeds the resources of

Mostowski's logic. This is the collective, as opposed to the usual, distributive use of quantifiers. Thus the sentence

(82) Five children ate the whole cake

cannot be formalized by

(83) $(!5x)(x$ is a child, x ate the whole cake),

which says that there are exactly five children each of which ate the whole cake. Collective and nonsortal quantifications will not be dwelt on in this book.

6 From Predicative to Relational Quantifiers

The generalized logic with 1- and 2-place quantifiers defined in the last section can easily be extended to a logic with n-place quantifiers for any positive integer n (Lindström, 1966). With each n-place quantifier Q^n we associate a family of cardinality functions t^{Q^n}, which, given a cardinal α, assigns a truth value to each 2^n-tuple of cardinals whose sum is α. Then, given a model $\mathfrak{A}$ with a universe A and a sequence of n subsets of A, $\langle B_1, \ldots, B_n\rangle$, the value of $Q^n_A(B_1, \ldots, B_n)$ depends on whether the atomic Boolean algebra generated by $B_1, \ldots, B_n$ in A is such that

$$t^{Q^n}_A(\beta_1, \ldots, \beta_{2^n}) = T,$$

where $\beta_1, \ldots, \beta_{2^n}$ are the cardinalities of all the atoms of this Boolean algebra ordered in some canonical manner. I will call the n-place quantifiers described above *predicative quantifiers* because such quantifiers constitute n-place relations among first-level (1-place) *predicates*.

The next step is to consider quantifiers on relations, or *relational quantifiers*. Syntactically, a 1-place relational quantifier is an operator that binds a formula by a sequence of n bound variables, $\langle x_1, \ldots, x_n\rangle$, for some finite $n > 1$. If we change the symbolization of 1- and 2-place predicative quantifiers to Q^1 and $Q^{1,1}$ respectively, we will naturally symbolize 1-place relational quantifiers in n variables by Q^n. Thus if

$$\Phi(x, y)$$

is a formula with x and y free, then

$$(Q^2x, y)\Phi(x, y)$$

is also a formula, generated from $\Phi(x, y)$ by binding the free variables x and y with Q^2. (The superscript "2" indicates that Q is a 1-place quantifier over 2-place first-level relations. For 2-place relational quantifiers over n- and m-place relations, in that order, we will use the superscript "n, m.")

Semantically, however, the characterization of logical relational quantifiers is an involved matter. The question is how to interpret the semantic condition (LQ2) with respect to these quantifiers. Recall that (LQ2) stipulates that quantifiers should not distinguish the identity of particular individuals in the universe of a given model. Mostowski construed this condition as requiring that quantifiers be invariant under permutations of the universe. But Mostowski dealt with predicative quantifiers, which semantically are functions on *subsets of the universe*, and the quantifiers we are dealing with now are relational quantifiers, i.e., functions on *subsets of Cartesian products of the universe*. If, following Mostowski, we again interpret (LQ2) as invariance under permutations, the question arises, invariance under permutations of what? Should relational quantifiers over a universe A, say 1-place quantifiers over binary relations on A, be invariant under permutations of A? Permutations of $A \times A$? Permutations of $A \times A$ induced in some specified manner by permutations of A? Or should we not interpret (LQ2) in terms of permutations of the universe at all when it comes to relational quantifiers? This question was raised by Higginbotham and May in "Questions, Quantifiers, and Crossing" (1981). From another angle Higginbotham and May ask what is implied by the requirement that quantifiers should not distinguish the *identity* of elements in the universe of discourse.

Yet another question is the relationship between logicality and cardinality. When I earlier discussed Mostowski's generalization, I said that this question could be avoided because on a very natural interpretation of (LQ2), the requirement that logical quantifiers not distinguish the identity of elements in the universe coincides with the requirement that logical quantifiers be definable by cardinality functions. Since (LQ2) is a natural condition on logical quantifiers, the identification of logical-predicative quantifiers with cardinality quantifiers appeared to be justified. However, now that the interpretation of (LQ2) is no longer straightforward, the question of cardinality and logicality has to be tackled directly.

But the question we have to confront first concerns (LQ2) itself. Why should (LQ2) be the semantic condition on logical quantifiers? Neither Mostowski nor Dummett (nor, as I have already indicated, Lindström) have justified their "choice" of invariance under permutations as the characteristic trait of logical quantifiers. So far I too have uncritically accepted their criterion. But in view of the questions we are now facing and in light of the general inquiry we have undertaken in this work, it is

now time to rethink the issue of logicality. Without a clear answer to the question of what makes a term *logical*, I doubt that we will be able to resolve the uncertainty regarding the correct definition of relational quantifiers. Moreover, a critical analysis of logicality will enable us to evaluate Mostowski's claim—most central to our query—that symbolic logic is *not exhausted* by standard mathematical first-order logic.

Chapter 3
To Be a Logical Term

Since the discovery of generalized quantifiers by A. Mostowski (1957), the question "What is a logical term?" has taken on a significance it did not have before. Are Mostowski's quantifiers "logical" quantifiers? Do they differ in any significant way from the standard existential and universal quantifiers? What logical operators, if any, has he left out? What, in all, are the first- and second-level predicates and relations that can be construed as logical?

One way in which I do not want to ask the question is, "What, *in the nature of things*, makes a property or a relation logical?" On this road lie the controversies regarding necessity and apriority, and these, I believe, should be left aside. Although some understanding of the modalities is essential for our enterprise, only their most general features come into play. A detailed study of complex and intricate modal and epistemic issues would just divert our attention and is of little use here. But if "the nature of things" is not our measure, what is? What should our starting point be? What strategy shall we decide upon?

A promising approach is suggested by L. Tharp in "Which Logic Is the Right Logic?" (1975). Tharp poses the question, What properties should a system of logic have? In particular, is standard first-order logic the "right" logic? To answer questions of this kind, he observes, it is crucial to have a clear idea about "the *role* logic is expected to play."[1] Tharp's point is worth taking, and it provides the clue we are searching for. If we identify a central role of logic and, relative to that role, ask what expressions can function as logical terms, we will have found a perspective that makes our question answerable, and significantly answerable at that.

The most suggestive discussion of the logical enterprise that I know of appears in A. Tarski's early papers on the foundations of semantics. Tarski's papers reveal the forces at work during the inception of modern

logic; at the same time, the principles developed by Tarski in the 1930s are still the principles underlying logic in the early 1990s. My interest in Tarski is, needless to say, not historical. I am interested in the modern conception of logic as it evolved out of Tarski's early work in semantics.

1 The Task of Logic and the Origins of Semantics

In "The Concept of Truth in Formalized Languages" (1933), "On the Concept of Logical Consequence" (1936a), and "The Establishment of Scientific Semantics" (1936b), Tarski describes the semantic project as comprising two tasks:

1. Definition of the *general* concept of truth for formalized languages
2. Definition of the *logical* concepts of truth, consequence, consistency, etc.

The main purpose of (1) is to secure metalogic against semantic paradoxes. Tarski worried lest the uncritical use of semantic concepts prior to his work concealed an inconsistency: a hidden fallacy would undermine the entire venture. He therefore sought precise, materially, as well as formally, correct definitions of "truth" and related notions to serve as a hedge against paradox. This aspect of Tarski's work is well known. In "Model Theory before 1945" R. Vaught (1974) puts Tarski's enterprise in a slightly different light:

> [During the late 1920s] Tarski had become dissatisfied with the notion of truth as it was being used. Since the notion "σ is true in $\mathfrak{A}$" is highly intuitive (and perfectly clear for any definite σ), it had been possible to go even as far as the completeness theorem by treating truth (consciously or unconsciously) essentially as an undefined notion—one with many obvious properties. ... But no one had made an analysis of truth, not even of exactly what is involved in treating it in the way just mentioned. At a time when it was quite well understood that 'all of mathematics' could be done, say, in ZF, with only the primitive notion $\in$, this meant that the theory of models (and hence much of metalogic) was indeed not part of mathematics. It seems clear that this whole state of affairs was bound to cause a lack of sure-footedness in metalogic. ... [Tarski's] major contribution was to show that the notion "σ is true in $\mathfrak{A}$" can simply be *defined* inside of ordinary mathematics, for example, in ZF.[2]

On both accounts the motivation for (1) has to do with the adequacy of the system designed to carry out the logical project, not with the logical project itself. The goal of logic is not the mathematical definition of "true sentence," and (1) is therefore a secondary, albeit crucially important, task of Tarskian logic. (2), on the other hand, does reflect Tarski's vision of the role of logic. In paper after paper throughout the early 1930s Tarski described the logical project as follows:[3] The goal is to develop and study deductive systems. Given a formal system $\mathscr{L}$ with language L and a definition of "meaningful," i.e., "well-formed," sentence for L, a (*closed*) *deductive system* in $\mathscr{L}$ is the set of all logical consequences of some set X of meaningful sentences of L. "Logical consequence" was defined proof-theoretically in terms of logical axioms and rules of inference: if $\mathscr{A}$ and $\mathscr{R}$ are the sets of logical axioms and rules of inference of $\mathscr{L}$, respectively, the set of *logical consequences of* X *in* $\mathscr{L}$ is the smallest set of well-formed sentences of L that includes X and $\mathscr{A}$ and is closed under the rules in $\mathscr{R}$. In contemporary terminology, a deductive system is a *formal theory* within a logical framework $\mathscr{L}$. (Note that the logical framework itself can be viewed as a deductive system, namely by taking X to be the set of logical axioms.) The task of logic, in this picture, is performed in two steps: (a) the construction of a logical framework for formal (formalized) theories; (b) the investigation of the logical properties—consistency, completeness, axiomatizability, etc.—of formal theories relative to the logical framework constructed in step (a). The concept of *logical consequence* (together with that of a well-formed formula) is the key concept of Tarskian logic. Once the definition of "logical consequence" is given, we can easily obtain not only the notion of a deductive system but also those of a logically true sentence; logically equivalent sets of sentences; an axiom system of a set of sentences; and axiomatizability, completeness, and consistency of a set of sentences. The study of the conditions under which various formal theories possess these properties forms the subject matter of metalogic.

Whence semantics? Prior to Tarski's "On the Concept of Logical Consequence" the definitions of the logical concepts were proof-theoretical. The need for semantic definitions of the same concepts arose when Tarski realized that there was a serious gap between the proof-theoretic definitions and the intuitive concepts they were intended to capture: many intuitive consequences of deductive systems could not be detected by the standard system of proof. Thus the sentence "For every natural number n, Pn" seems to follow, in some important sense, from the set of sentences "Pn," where n is a natural number, but there is no way to express this fact by the proof method for standard first-order logic. This situation, Tarski said, shows that proof theory by itself cannot fully accomplish the task of logic. One might contemplate extending the system by adding new rules of inference, but to no avail. Gödel's discovery of the incompleteness of the deductive system of Peano arithmetic showed,

> In every deductive theory (apart from certain theories of a particularly elementary nature), however much we supplement the ordinary rules of inference by new purely structural rules, it is possible to construct sentences which follow, in the usual sense, from the theorems of this theory, but which nevertheless cannot be proved in this theory on the basis of the accepted rules of inference.[4]

Tarski's conclusion was that proof theory provides only a partial account of the logical concepts. A new method is called for that will permit a more comprehensive systematization of the intuitive content of these concepts.

The intuitions underlying our informal notion of logical consequence (and derivative concepts) are anchored, according to Tarski, in certain relationships between linguistic items and objects in (configurations of) the world. The discipline that studies relationships of this kind is called *semantics*:

> We . . . understand by semantics the totality of considerations concerning those concepts which, roughly speaking, express certain connexions between the expressions of a language and the objects and states of affairs referred to by these expressions.[5]

The precise formulation of the intuitive content of the logical concepts is therefore a job for semantics. (Although the relation between the set of sentences "Pn" and the universal quantification "$(\forall x)Px$," where x ranges over the natural numbers and "n" stands for a name of a natural number, is not logical consequence, we will be able to characterize it accurately within the framework of Tarskian semantics, e.g., in terms of ω completeness.)

2 The Semantic Definition of "Logical Consequence" and the Emergence of Models

Tarski describes the intuitive content of the concept "logical consequence" as follows:

> Certain considerations of an intuitive nature will form our starting-point. Consider any class K of sentences and a sentence X which follows from the sentences of this class. From an intuitive standpoint it can never happen that both the class K consists only of true sentences and the sentence X is false. Moreover, . . . we are concerned here with the concept of logical, i.e. *formal*, consequence, and thus with a relation which is to be uniquely determined by the form of the sentences between which it holds. . . . The two circumstances just indicated . . . seem to be very characteristic and essential for the proper concept of consequence.[6]

We can express the two conditions set by Tarski on a correct definition of "logical consequence" by (C1) and (C2) below:

CONDITION C1 If X is a logical consequence of K, then X is a *necessary* consequence of K in the following intuitive sense: it is impossible that all the sentences of K are true and X is false.

CONDITION C2 Not all necessary consequences fall under the concept of logical consequence; only those in which the consequence relation between a set of sentences K and a sentence X is based on *formal* relationships between the sentences of K and X do.

To provide a formal definition of "logical consequence" based on (C1) and (C2), Tarski introduces the notion of *model*. In current terminology, given a formal system $\mathscr{L}$ with a language L, an *$\mathscr{L}$-model*, or a *model for* $\mathscr{L}$, is a pair, $\mathfrak{A} = \langle A, D\rangle$, where A is a set and D is a function that assigns to the nonlogical primitive constants of L, $t_1, t_2, \ldots$ elements (or constructs of elements) in A: if t_i is an individual constant, $D(t_i)$ is a member of A; if t_i is an n-place first-level predicate, $D(t_i)$ is an n-place relation included in A^n; etc. We will say that the function D assigns to $t_1, t_2, \ldots$ denotations in A. Any pair of a set A and a denotation function D determines a model for $\mathscr{L}$. Given a theory T in a formal system $\mathscr{L}$ with a language L, we say that a model $\mathfrak{A}$ for $\mathscr{L}$ is a *model of* $\mathscr{T}$ iff every sentence of $\mathscr{T}$ is true in $\mathfrak{A}$. (Similarly, $\mathfrak{A}$ is a model of a sentence X of L iff X is true in $\mathfrak{A}$.) The definition of "the sentence X of L is true in a model $\mathfrak{A}$ for $\mathscr{L}$" is given in terms of satisfaction: X is *true in* $\mathfrak{A}$ iff every assignment of elements in A to the variables of L satisfies X in $\mathfrak{A}$. The notion of satisfaction is based on Tarski 1933. I assume that the reader is familiar with this notion.

The formal definition of "logical consequence" in terms of models proposed by Tarski is:

DEFINITION LC The sentence X *follows logically* from the sentences of the class K iff every model of the class K is also a model of the sentence X.[7]

The definition of "logical truth" immediately follows:

DEFINITION LTR The sentence X is *logically true* iff every model is a model of X.

To be more precise, (LC) and (LTR) should be relativized to a logical system $\mathscr{L}$. "Sentence" would then be replaced by "$\mathscr{L}$-sentence" and "model" by "$\mathscr{L}$-model."

(A historical remark is in place here. Some philosophers claim that Tarski's 1936 definition of a model is essentially different from the one currently used because in 1936 Tarski did not require that models vary

with respect to their universes. This issue does not really concern us here, since we are interested in the legacy of Tarski, not this or that historical stage in the development of his thought. For the intuitive ideas we go to the early writings, where they are most explicit, while the formal constructions are those that appear in his mature work.

Notwithstanding the above, it seems to me highly unlikely that in 1936 Tarski intended all models to share the same universe. This is because such a notion of model is incompatible with the most important model-theoretic results obtained by logicians, including Tarski himself, before that time. Thus, the Löwenheim-Skolem-Tarski theorem (1915–1928) says that if a first-order theory has a model with an infinite universe A, it has a model with a universe of cardinality α for every infinite α. Obviously, this theorem does not hold if one universe is common to all models. Similarly, Gödel's 1930 completeness theorem fails: if all models share the same universe, then for every positive integer n, one of the two first-order statements "There are more than n things" and "There are at most n things" is true in all models, and hence, according to (LTR), it is logically true. But no such statement is provable from the logical axioms of standard first-order logic.[8] Be that as it may, the Tarskian concept of model discussed here does include the requirement that any nonempty set is the universe of some model for the given language.)

Does (LC) satisfy the intuitive requirements on a correct definition of "logical consequence" given by (C1) and (C2) above? According to Tarski it does:

> It seems to me that everyone who understands the content of the above definition must admit that it agrees quite well with common usage. ... It can be proved, on the basis of this definition, that every consequence of true sentences must be true, and also that the consequence relation which holds between given sentences is completely independent of the sense of the extra-logical constants which occur in these sentences.[9]

In what way does (LC) satisfy (C1)? Tarski mentions the existence of a proof but does not provide a reference. There is a very simple argument that, I believe, is in the spirit of Tarski:[10]

Proof Assume X is a logical consequence of K, i.e., X is true in all models in which all the members of K are true. Suppose that X is not a necessary consequence of K. Then it is possible that all the members of K are true and X is false. But in that case there is a model in which all the members of K come out true and X comes out false. Contradiction.

The argument is simple. However, it is based on a crucial assumption:

ASSUMPTION AS If K is a set of sentences and X is a sentence (of a formal language L of $\mathscr{L}$) such that it is intuitively possible that all the members of K are true while X is false, then there is a model (for $\mathscr{L}$) in which all the members of K come out true and X comes out false.

Assumption (AS) is equivalent to the requirement that, given a logic $\mathscr{L}$ with a formal language L, every possible state of affairs relative to the expressive power of L be represented by some model for $\mathscr{L}$. (Note that (AS) does not entail that every state of affairs represented by a model for $\mathscr{L}$ is possible. This accords with Tarski's view that the notion of *logical possibility* is weaker than, and hence different from, the general notion of possibility [see (C2)].) Is (AS) fulfilled by Tarski's model-theoretic semantics?

We can show that (AS) holds at least for standard first-order logic. Let $\mathscr{L}$ be a standard first-order system, L the language of $\mathscr{L}$, K a set of sentences of L, and X a sentence of L. Suppose it is intuitively possible that all the members of K are true and X is false. Then, if we presume that the rules of inference of standard first-order logic are necessarily truth-preserving, $K \cup \{\sim X\}$ is intuitively consistent in the proof-theoretic sense: for no first-order sentence Y are both Y and $\sim Y$ provable from $K \cup \{\sim X\}$. It follows from the completeness theorem for first-order logic that there is a model for $\mathscr{L}$ in which all the sentences of K are true and X is false.

As for (C2), Tarski characterizes the formality requirement as follows:

> Since we are concerned here with the concept of logical, i.e., *formal* consequence, and thus with a relation which is to be uniquely determined by the form of the sentences between which it holds, this relation cannot be influenced in any way by empirical knowledge, and in particular by knowledge of the objects to which the sentence X or the sentences of the class K refer. The consequence relation cannot be affected by replacing the designations of the objects referred to in these sentences by the designations of any other objects.[11]

The condition of formality, (C2), has several aspects. First, logical consequences, according to Tarski, are based on the logical form of the sentences involved. The logical form of sentences is in turn determined by their logical terms (see Tarski's notion of a well-formed formula in "The Concept of Truth in Formalized Languages"). Therefore, logical consequences are based on the logical terms of the language. Second, logical consequences are not empirical. This means that logical terms, which determine logical consequences, are not empirical either. Finally, logical consequences "cannot be affected by replacing the designations of the objects... by other objects." In "The Concept of Logical Consequence" Tarski first attempted a substitutional interpretation of the last requirement. This led to a substitutional definition of "logical consequence." According to this definition, consequences preserved under all (uniform, type preserving) substitutions of the nonlogical terms of the language are logical. However, Tarski soon realized that the substitutional definition did not capture the notion of logical consequence in all its generality.[12] The substitutional test depends on the expressive power of the language in question. In particular, languages with a meager vocabulary of singular terms let intuitively nonlogical consequences pass for genuinely logical ones. Tarski's reaction to the shortcomings of the substitutional test was to drop the idea of substitutivity altogether. Instead, Tarski turned to *semantics*, a new discipline devoted to studying the relation between language and the world, whose basic notions are "satisfaction" and "model." On the basis of these concepts Tarski proposed the model-theoretic definition of logical consequence, (LC). Although Tarski did not explain what "indifference of the consequence relation to replacement of objects" meant semantically, I think we can offer the following analysis inspired by Mostowski. There are terms that take the identity of objects into account and terms that do not. Terms underlying logical consequences must be of the second kind. That is to say, logical terms should not distinguish the identity of objects in the universe of any model. (By "identity of an object" I here mean the features that make an object what it is, the properties that single it out.)

Now clearly Tarskian consequences of standard first-order logic satisfy the formality condition. First, only entirely trivial consequences (X follows logically from K just in case $X \in K$) obtain without logical terms. Therefore, logical consequences are due to logical terms of the language. Second, the truth-functional connectives, identity, and the universal and existential quantifiers are nonempirical functions that do not distinguish the objects in any given model. The substitution test, which is still necessary (though not sufficient), is also passed by standard logic.

We see that (C2), the condition of formality, sets a limit on (C1), the condition of necessity: necessity does not suffice for logicality. While all logical consequences are necessary, only necessary consequences that are also formal count as genuinely logical. An example of a necessary consequence that fails to satisfy the condition of formality is,

(1) b is red all over; therefore b is not blue all over.

This consequence is not logical according to Tarski's criterion, because it hangs on particular features of color properties that depend on the identity of objects in the universe of discourse. (Try to replace "blue" with "smooth," a replacement that has no bearing on the formal relations between premise and conclusion, and see what happens.) Later we will also see that (C1) sets a restriction on the application of (C2).

I think conditions (C1) and (C2) on the key concept of logical consequence delineate the scope as well as the limit of Tarski's enterprise: the development of a conceptual system in which the concept of logical consequence ranges over *all* formally necessary consequences and nothing else. Since our intuitions leave some consequences undetermined with respect to formal necessity, the boundary of the enterprise is somewhat vague. But the extent of vagueness is limited. Formal necessity is a relatively unproblematic notion, and the persistent controversies involving the modalities are not centered around the formal.

We have seen that at least in one application, namely, in standard first-order logic, Tarski's definition of logical consequence stands the test of (C1) and (C2): all the standard consequences that fall under Tarski's definition are indeed formal and necessary. We now ask, Does standard first-order logic yield *all* the formally necessary consequences with a first-level (extensional) vocabulary? Could not the standard system be extended so that Tarski's definition encompasses new consequences satisfying the intuitive conditions but undetected within the standard system? Tarski himself all but asked the same question. He ended "On the Concept of Logical Consequence" with the following note:

> Underlying our whole construction is the division of all terms of the language discussed into logical and extra-logical. This division is certainly not quite arbitrary. If, for example, we were to include among the extra-logical signs the implication sign, or the universal quantifier, then our definition of the concept of consequence would lead to results which obviously contradict ordinary usage. On the other hand no objective grounds are known to me which permit us to draw a sharp boundary between the two groups of terms. It seems to be possible to include among logical terms some which are usually regarded by logicians as extra-logical without running into consequences which stand in sharp contrast to ordinary usage.[13]

The question, "What is the full scope of logic?" I will ask in the form: What is the widest notion of a *logical term* for which the Tarskian definition of "logical consequence" gives results compatible with (C1) and (C2)?

3 Logical and Extralogical Terms: An Unfounded Distinction?

What is the widest definition of "logical term" compatible with Tarski's theory? In 1936 Tarski did not know how to handle the problem of new logical terms. Tarski's interest was not in extending the scope of "logical

consequence" but in defining this concept successfully for standard logic. From this point of view, the relativization of "logical consequence" to collections of logical terms was disquieting. While Tarski's definition produced the right results when applied to standard first-order logic, there was no guarantee that it would continue to do so in the context of wider "logics." A standard for logical terms could solve the problem, but Tarski had no assurance that such a standard was to be found. The view that Tarski's notion of logical consequence is inevitably tied up with arbitrary choices of logical terms was advanced by J. Etchemendy (1983, 1990). Etchemendy was quick to point out that this arbitrary relativity undermines Tarski's theory. I will not discuss Etchemendy's interpretation of Tarski here, but I would like to examine the issue in the context of my own analysis. Is the distinction between logical and extralogical terms founded? If it is, what is it founded on? Which term falls under which category?

Tarski did not see where to draw the line. In 1936 he went as far as saying that "in the extreme case we could regard all terms of the language as logical. The concept of *formal* consequence would then coincide with that of *material* consequence."[14] Unlike "logical consequence," the concept of material consequence is defined without reference to models:

DEFINITION MC The sentence X is a *material consequence* of the sentences of the class K iff at least one sentence of K is false or X is true.[15]

Tarski's statement first seemed to me clear and obvious. However, on second thought I found it somewhat puzzling. How could *all* material consequences of a hypothetical first-order logic $\mathscr{L}$ become logical consequences? Suppose $\mathscr{L}$ is a logic in which "all terms are regarded as logical." Then evidently the standard logical constants are also regarded as logical in $\mathscr{L}$. Consider the $\mathscr{L}$-sentence:

(2) There is exactly one thing,

or, formally,

(3) $(\exists x)(\forall y)x = y$.

This sentence is false in the real world, hence

(4) There are exactly two things

follows materially from it (in $\mathscr{L}$). But Tarski's semantics demands that for each cardinality α, there be a model for $\mathscr{L}$ with a universe of cardinality α. (This much comes from his requirement that any arbitrary set of objects constitute the universe of some model for $\mathscr{L}$). Thus in particular $\mathscr{L}$ has a model with exactly one individual. It is therefore not true that in every model in which (2) is true, (4) is true too. Hence, according to Tarski's definition, (4) is not a logical consequence of (2).

So Tarski conceded too much: no addition of new logical terms would trivialize his definition altogether. Tarski underestimated the viability of his system. His model-theoretic semantics has a built-in barrier that prevents a total collapse of logical into material consequence. To turn all material consequences of a given formal system $\mathscr{L}$ into logical consequences requires limiting the totality of sets in which $\mathscr{L}$ is to be interpreted. But the requirement that no such limit be set is intrinsic to Tarski's notion of a model.

It appears, then, that what Tarski had to worry about was not total but partial collapse of logical into material consequence. However, it is still not clear what "regarding all the terms of the language as logical" meant. Surely Tarski did not intend to say that if all the constant terms of a logic $\mathscr{L}$ are logical, the distinction between formal and material consequence for $\mathscr{L}$ collapses. The language of pure identity is a conspicuous counterexample. All the constant terms of that language are logical, yet the definition of "logical consequence" yields a set of consequences different (in the right way) from the set of material consequences.

We should also remember that Tarski's definition of "logical consequence" and the definition of "satisfaction" on which it is based are applicable only to formalized languages whose vocabulary is essentially restricted. Therefore, Tarski could not have said that if we regard all terms of natural language as logical, the definition of "logical consequence" will coincide with that of "material consequence". A circumstance concerning natural language in its totality could not have any effect on the Tarskian concept of logical consequence.

Even with respect to single constants it is not altogether clear what treating them as logical might mean. Take, for instance, the term "red." How do you construe "red" as a logical constant? To answer this question we have to find out what makes a term logical (extralogical) in Tarski's system. Only then will we be able to determine whether any term whatsoever can be regarded as logical in Tarski's logic.

4 The Roles of Logical and Extralogical Terms

What makes a term logical or extralogical in Tarski's system? Considering the question from the "functional" point of view I have opted for, I ask: How does the dual system of a formal language and its model-theoretic semantics accomplish the task of logic? In particular, what is the role

of logical and extralogical constants in determining logical truths and consequences?

Extralogical constants

Consider the statement

(5) Some horses are white,

formalized in standard first-order logic by

(6) $(\exists x)(Hx \mathbin{\&} Wx)$.

How does Tarski succeed in giving this statement truth conditions that, in accordance with our clear pretheoretical intuitions, render it logically indeterminate (i.e., neither logically true nor logically false)? The crucial point is that the common noun "horse" and the adjective "white" are interpreted within models in such a way that their intersection is empty in some models and not empty in others. Similarly, for any natural number n, the sentence

(7) There are n white horses

is logically indeterminate because in some but not all models "horse" and "white" are so interpreted as to make their intersection of cardinality n. Were "finitely many" expressible in the logic, a similar configuration would make

(8) Finitely many horses are white

logically indeterminate as well.

In short, what is special to extralogical terms like "horse" and "white" in Tarskian logic is their *strong semantic variability*. Extralogical terms have no independent meaning: they are interpreted only *within* models. Their meaning in a given model is nothing more than the value that the denotation function D assigns to them in that model. We cannot speak about *the* meaning of an extralogical term: being extralogical implies that nothing is ruled out with respect to such a term. Every denotation of the extralogical terms that accords with their syntactic category appears in some model. Hence the totality of interpretations of any given extralogical term in the class of all models for the formal system is exactly the same as that of any other extralogical term of the same syntactic category. Since every set of objects is the universe of some model, any possible state of affairs — any possible configuration of individuals, properties, relations, and functions — via-à-vis the extralogical terms of a given formalized language (possible, that is, with respect to their meaning prior to formalization) is represented by some model.

Formally, we can define Tarskian extralogical terms as follows:

DEFINITION ET $\{e_1, e_2, \ldots\}$ is the set of primitive *extralogical terms* of a Tarskian logic $\mathscr{L}$ iff for every set A and every function D that assigns to $e_1, e_2, \ldots$ denotations in A (in accordance with their syntactic categories), there is a model $\mathfrak{A}$ for $\mathscr{L}$ such that $\mathfrak{A} = \langle A, D\rangle$.

It follows from (ET) that primitive extralogical terms are semantically unrelated to one another. As a result, complex extralogical terms, produced by intersections, unions, etc. of primitive extralogical terms (e.g., "horse and white") are strongly variable as well.

Note that it is essential to take into account the strong variability of extralogical terms in order to understand the meaning of various claims of logicality. Consider, for instance, the statement

(9) $(\exists x)x =$ Jean-Paul Sartre,

which is logically true in a Tarskian logic with "Jean-Paul Sartre" as an extralogical individual constant. Does the claim that (9) is logically true mean that the existence (unspecified with respect to time) of the deceased French philosopher is a matter of logic? Obviously not. The logical truth of (9) reflects the principle that if a term is used in a language to name objects, then in every model for the language some object is named by that term. But since "Jean-Paul Sartre" is a strongly variable term, what (9) says is "There is *a* Jean-Paul Sartre," not "*The* (French philosopher) Jean-Paul Sartre exists."

Logical constants

It has been said that to be a logical constant in a Tarskian logic is to have *the same* interpretation in all models. Thus for "red" to be a logical constant in logic $\mathscr{L}$, it has to have a constant interpretation in all the models for $\mathscr{L}$. I think this characterization is faulty because it is vague. How do you interpret "red" *in the same way* in all models? "In the same way" in what sense? Do you require that in every model there be the same number of objects falling under "red"? But for every number larger than 1 there is a model that cannot satisfy this requirement simply because it does not have enough elements. So at least in one way, cardinalitywise, the interpretation of "red" must vary from model to model.

The same thing holds for the standard logical constants of Tarskian logic. Take the universal quantifier. In every model for a first-order logic the universal quantifier is interpreted as a singleton set (i.e., the set of the universe).[16] But in a model with 10 elements it is a set of a set with 10 elements, whereas in a model with 9 elements it is a set of a set with 9 elements. Are these interpretations the same?[17]

I think that what distinguishes logical constants in Tarski's semantics is not the fact that their interpretation does not vary from model to model (it does!) but the fact that they are interpreted *outside* the system of models.[18] The meaning of a logical constant is not given by the definitions of particular models but is part of the same metatheoretical machinery used to define the entire network of models. The meaning of logical constants is given by *rules external to the system*, and it is due to the existence of such rules that Tarski could give his recursive definition of truth (satisfaction) for well-formed formulas of any given language of the logic. Syntactically, the logical constants are "fixed parameters" in the inductive definition of the set of well-formed formulas; semantically, the rules for the logical constants are the functions on which the definition of satisfaction by recursion (on the inductive structure of the set of well-formed formulas) is based.

How would different choices of logical terms affect the extension of "logical consequence"? Well, if we contract the standard set of logical terms, some intuitively formal and necessary consequences (i.e., certain logical consequences of standard first-order logic) will turn nonlogical. If, on the other hand, we take any term whatsoever as logical, we will end up with new "logical" consequences that are intuitively not formally necessary. The first case does not require much elaboration: if "and" were interpreted as "or," X would not be a logical consequence of "X and Y." As for the second case, let us take an extreme example. Consider the natural-language terms "Jean-Paul Sartre" and "accepted the Nobel Prize in literature," and suppose we use them as logical terms in a Tarskian logic by keeping their usual denotation "fixed." That is, the semantic counterpart of "Jean-Paul Sartre" will be the existentialist French philosopher Jean-Paul Sartre, and the semantic counterpart of "accepted the Nobel Prize in literature" will be the set of all actual persons up to the present who (were awarded and) accepted the Nobel Prize in literature. Then

(10) Jean-Paul Sartre accepted the Nobel Prize in literature

will come out false, according to Tarski's rules of truth (satisfaction), no matter what model we are considering. This is because, when determining the truth of (10) in any given model $\mathfrak{A}$ for the logic, we do not have to look in $\mathfrak{A}$ at all. Instead, we examine two fixed entities outside the apparatus of models and determine whether the one is a member of the other. This renders (10) logically false, and according to Tarski's definition, any sentence of the language we are considering follows logically from it, in contradiction with the pretheoretical conditions (C1) and (C2).

The above example violates two principles of Tarskian semantics: (1) "Jean-Paul Sartre" and "accepted the Nobel Prize in literature" do not satisfy the requirement of formality. (2) The truth conditions for (10) bypass the very device that serves in Tarskian semantics to distinguish material from logical consequence, namely the apparatus of models. No wonder the definition of "logical consequence" fails!

It is easy to see that each violation by itself suffices to undermine Tarski's definition. In the case of (1), "Jean-Paul Sartre" and "accepted the Nobel Prize in literature" are empirical terms that do distinguish between different objects in the universe of discourse. As for (2), suppose we define logical terms in accordance with (C2) but without reference to the totality of models. Say we interpret the universal quantifier for a single universe, that of the natural numbers. In that case "for every" becomes "for every natural number," and the statement

(11) Every object is different from at least three other objects

turns out logically true, in violation of the intuition embedded in (C1). By requiring that "every" be defined over all models, we circumvent the undesirable result.

We can now see how Tarski's method allows us to identify a sentence like

(12) Everything is identical with itself

as the logical truth that it intuitively is. The crucial point is that the intuitive meanings of "is identical with" and "everything" are captured by rules definable over all models. These rules single out pairs and sets of objects that share certain formal features which do not vary from one possible state of affairs to another. Thus in all models (representations of possible states of affairs), the set of self-identical objects is universal (i.e., coincides with the universe), and in each model the universal set is "everything" for that particular model.

5 The Distinction between Logical and Extralogical Terms: A Foundation

The discussion of logical and extralogical terms enables us to answer the questions posed in section 3. We understand what it means to regard *all*

terms of the language as logical. Within the scheme of Tarski's logic it means to allow any rule whatsoever to be the semantic definition of a logical constant. In particular, the intuitive interpretation of any term becomes its semantic rule qua a logical term. Our investigation clearly demonstrated that not every interpretation of logical terms is compatible with Tarski's vision of the task of logic.

We can now turn to the main question of section 3. Is the distinction between logical and extralogical terms founded? Of course it is! The distinction between logical and extralogical terms is founded on our pretheoretical intuition that logical consequences are distinguished from material consequences in being necessary and formal. To reject this intuition is to drop the foundation of Tarski's logic. To accept it is to provide a ground for the division of terms into logical and extralogical.

But what is the boundary between logical and extralogical terms? Should we simply say that a constant is logical if adding it to the standard system would not conflict with (C1) and (C2)? This criterion is correct but not very informative. It appears that consequences like

(13) Exactly one French philosopher refused the Nobel Prize in literature; therefore, finitely many French philosophers did

are formal and necessary in Tarski's sense. Therefore "finitely many" is a reasonable candidate for logical constanthood. But can we be sure that "finitely many" will never lead to a conflict with (C1) and (C2)? And will our intuitions guide us in each particular case? By themselves, (C1) and (C2) do not provide a usable criterion. Let us see if their analysis in the context of Tarski's system will not lead us to the desired criterion.

The view that logic is an instrument for identifying formal and necessary consequences leads to two initial requirements (based on (C1) and (C2)): (1) that every possible state of affairs vis-à-vis a given language be represented by some model for the language, and (2) that logical terms represent formal features of possible states of affairs, i.e., formal properties of (relations among) constituents of states of affairs. To satisfy these requirements the Tarskian logician constructs a dual system, each part of which is itself a complex, syntactic-semantic structure. One constituent includes the extralogical vocabulary (syntax) and the apparatus of models (semantics). I will call it the *base* of the logic. (Note that only extralogical terms, not logical terms, play a role in constructing models.) In a first-order logic the base is strictly first-level: syntactically, the extralogical vocabulary includes only singular terms and terms whose arguments are singular; semantically, in any given model the extralogical terms are assigned only individuals or sets, relations, and functions of individuals.

The second part consists of the logical terms and their semantic definitions. Its task is to introduce formal structure into the system. Syntactically, logical terms are formula-building operators; semantically, they are assigned pre-fixed functions on models that express formal properties of, relations among, and functions of "elements of models" (objects in the universe and constructs of these). Since logical terms are meant to represent formal properties of elements of models corresponding to the extralogical vocabulary, their level is generally higher than that of nonlogical terms. Thus in standard first-order logic, identity is the only first-level logical term. The universal and existential quantifiers are second level, semantically as well as syntactically, and the logical connectives too are of higher level. As for singular terms, these can never be construed as logical. This is because singular terms represent atomic components of models, and atomic components, being atomic, have no structure (formal or informal). I will say that the system of logical terms constitutes a *superstructure* for the logic.

The whole system is brought together by superimposing the logical apparatus on the nonlogical base. Syntactically, this is done by rules for forming well-formed formulas by means of the logical operators, and semantically, by rules for determining truth (satisfaction) in a model based on the formal denotations of the logical vocabulary. (Note that since the systems we are considering are extensional, "interpretation" has the same import as "denotation.")

Now, to satisfy the conditions (C1) and (C2), it is essential that no logical term represent a property or a relation that is intuitively variable from one state of affairs to another. Furthermore, it is important that logical terms be formal entities. Finally, the denotations of logical terms need to be defined over models, all models, so that every possible state of affairs is taken into account in determining logical truths and consequences.

It appears that if we can specify a series of conditions that are exclusively and exhaustively satisfied by terms fulfilling the requirements above, we will have succeeded in defining "logical term" in accordance with Tarski's basic principles. In particular, the Tarskian definition of "logical consequence" (and the other metalogical concepts) will give correct results, all the correct results, in agreement with (C1) and (C2).

6 A Criterion for Logical Terms

My central idea is this. Logical terms are formal in a sense that was specified in section 2. There we already interpreted the requirement of formality in the spirit of Mostowski as "not distinguishing the identity of objects in a given universe." Why don't we take another step in the same direction and follow Mostowski's construal of "not distinguishing the identity of objects" as invariance under permutations (see chapter 2). Generalizing Mostowski, we arrive at the notion of a logical term as formal in the following sense: being formal is, semantically, being invariant under all nonstructural variations of models. That is to say, being formal is being invariant under isomorphic structures. In short, logical terms are *formal* in the sense of being essentially *mathematical*. Since, intuitively, the mathematical parameters of reality do not vary from one possible state of affairs to another, the claim that logical consequences are intuitively necessary is in principle satisfied by logics that allow mathematical terms as logical terms. My thesis, therefore, is this: all and only formal terms, terms invariant under isomorphic structures, can serve as logical terms in a logic based on Tarski's ideas. I must, however, add the proviso that new terms be incorporated in the logical system "in the right way."

I will now proceed to set down in detail the criterion for logical terms. But first let me make a few preliminary remarks. My analysis of Tarski's syntactic-semantic system did not depend on the particulars of the metatheoretic language in which the syntax and the semantics are embedded. In standard mathematical logic the metalanguage consists of a fragment of natural language augmented by first-order set theory or higher-order logic. In particular, models are set-theoretic constructs, and the definition of "satisfaction in a model" is accordingly set-theoretical. This feature of contemporary metalogic is, however, not inherent in the nature of the logical enterprise, and one could contemplate a background language different from the one currently used. Without committing myself to any particular metatheoretical mathematics, I will nevertheless use the terminology of standard first-order set theory in the formal entries of the criterion for logical terms, as this will contribute to precision and clarity.

For transparency I will not include sentential connectives in the criterion. While it is technically easy to construe the connectives as quantifiers (see Lindström 1966), the syntactic-semantic apparatus of Tarskian logic is superfluous for analyzing their scope. The standard framework of sentential logic is perfectly adequate, and relative to this framework, the problem of identifying all the logical connectives that there are has already been solved. The solution clearly satisfies Tarski's requirements: the standard logic of sentential connectives has a base that consists syntactically of extralogical sentential letters and semantically of a list of all possible assignments of truth values to these letters. Any possible state of affairs vis-à-vis the sentential language is represented by some assignment. The logical superstructure includes the truth-functional connectives and their semantic definitions. The connectives are both syntactically and semantically of a higher level than the sentential letters. Their semantic definitions are pre-fixed: logical connectives are semantically identified with truth-functional operators, and the latter are defined by formal (Boolean) functions whose values and arguments, i.e., truth values and sequences of truth values, represent possible states of affairs. This ensures that truths and consequences that hold in all "models" are formally necessary in Tarski's sense.

As for modal operators, they too are outside the scope of this investigation, though for different reasons. First, my criterion for logical terms is based on analysis of the Tarskian framework, which is insufficient for modals. Second, we cannot take it for granted that the task of modal logic is the same as that of symbolic logic proper. To determine the scope of modal logic and characterize its operators, we would have to set upon an independent inquiry into its underlying goals and principles.

Conditions on logical constants in first-order logics

The criterion for logical terms based on the Tarskian conception of formal first-order logic will be formulated in a series of individually necessary and collectively sufficient conditions. These conditions will specify what simple and/or complex terms from an initial pool of constants can serve as logical constants in a first-order logic. In stating these conditions, I place a higher value on clarity of ideas than on economy. As a result the conditions are not mutually independent.

A. A logical constant C is syntactically an n-place predicate or functor (functional expression) of level 1 or 2, n being a positive integer.

B. A logical constant C is defined by a single extensional function and is identified with its extension.

C. A logical constant C is defined over models. In each model $\mathfrak{A}$ over which it is defined, C is assigned a construct of elements of $\mathfrak{A}$ corresponding to its syntactic category. Specifically, I require that C be defined by a function f_c such that given a model $\mathfrak{A}$ (with universe A) in its domain:

   a. If C is a first-level n-place predicate, then $f_c(\mathfrak{A})$ is a subset of A^n.

   b. If C is a first-level n-place functor, then $f_c(\mathfrak{A})$ is a function from A^n into A.

   c. If C is a second-level n-place predicate, then $f_c(\mathfrak{A})$ is a subset of $B_1 \times \cdots \times B_n$, where for $1 \leq i \leq n$,

$$B_i = \begin{cases} A & \text{if } i(C) \text{ is an individual} \\ P(A^m) & \text{if } i(C) \text{ is an } m\text{-place predicate} \end{cases}$$

   ($i(C)$ being the ith argument of C).

   d. If C is a second-level n-place functor, then $f_c(\mathfrak{A})$ is a function from $B_1 \times \cdots \times B_n$ into B_{n+1}, where for $1 \leq i \leq n+1$, B_i is defined as in (c).

D. A logical constant C is defined over *all* models (for the logic).

E. A logical constant C is defined by a function f_c which is invariant under isomorphic structures. That is, the following conditions hold:

   a. If C is a first-level n-place predicate, $\mathfrak{A}$ and $\mathfrak{A}'$ are models with universes A and A' respectively, $\langle b_1, \ldots, b_n \rangle \in A^n$, $\langle b'_1, \ldots, b'_n \rangle \in A'^n$, and the structures $\langle A, \langle b_1, \ldots, b_n \rangle\rangle$ and $\langle A', \langle b'_1, \ldots, b'_n \rangle\rangle$ are isomorphic, then $\langle b_1, \ldots, b_n \rangle \in f_c(\mathfrak{A})$ iff $\langle b'_1, \ldots, b'_n \rangle \in f_c(\mathfrak{A}')$.

   b. If C is a second-level n-place predicate, $\mathfrak{A}$ and $\mathfrak{A}'$ are models with universes A and A' respectively, $\langle D_1, \ldots, D_n \rangle \in B_1 \times \cdots \times B_n$, $\langle D'_1, \ldots, D'_n \rangle \in B'_1 \times \cdots \times B'_n$ (where for $1 \leq i \leq n$, B_i and B'_i are as in (C.c)), and the structures $\langle A, \langle D_1, \ldots, D_n \rangle\rangle$, $\langle A', \langle D'_1, \ldots, D'_n \rangle\rangle$ are isomorphic, then $\langle D_1, \ldots, D_n \rangle \in f_c(\mathfrak{A})$ iff $\langle D'_1, \ldots, D'_n \rangle \in f_c(\mathfrak{A}')$.

   c. Analogously for functors.

Some explanations are in order. Condition (A) reflects the perception of logical terms as structural components of the language. In particular, it rules out individual constants as logical terms. Note, however, that although an individual by itself cannot be represented by a logical term (since it lacks "inner" structure), it can combine with functions, sets, or relations to form a structure representable by a logical term. Thus, below I define a logical constant that represents the structure of the natural numbers with their ordering relation and zero (taken as an individual). The upper limit on the level of logical terms is 2, since the logic we are considering is a logic for first-level languages, and a first-level language can only provide its logical terms with arguments of level 0 or 1.

Condition (B) ensures that logical terms are *rigid*. Each logical term has a *pre-fixed* meaning in the metalanguage. This meaning is unchangeable and is completely exhausted by its semantic definition. That is to say, from the point of view of Tarskian logic, there are no "possible worlds" of logical terms. Thus, qua logical terms, the expressions "the number of planets" and "9" are indistinguishable. If you want to express the intuition that the number of planets changes from one possible "world" to another, you have to construe it as an extralogical term. If, on the other hand, you choose to use it as a logical term (or in the definition of a logical term), only its extension counts, and this is the same as the extension of "9."

Condition (C) provides the tie between logical terms and the apparatus of models. By requiring that logical terms be defined by fixed functions from models to structures within models, it allows logical terms to represent "fixed" parameters of changeable states of affairs. By requiring that logical terms be defined for each model by elements of this model, it ensures that the apparatus of models is not bypassed when logical truths and consequences are determined. Condition (C) also takes care of the correspondence in categories between the syntax and the semantics.

The point of (D) is to ensure that *all* possible states of affairs are taken into account in determining logical truths and consequences. Thus truth-in-all-models is *necessary* truth and consequence-in-all-models is *necessary* consequence. Conditions (B) to (D) together express the requirement that logical terms are semantically superimposed on the apparatus of models.

With (E) I provide a general characterization of formality: to be *formal* is *not to distinguish between (to be invariant under) isomorphic structures*. This criterion is almost universally accepted as capturing the intuitive (semantic) idea of formality. I will trace the origins of condition (E) and discuss its significance separately in section 7 below. It follows from (E) that if $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are models with the same universe A, then for every logical term C, $f_C(\mathfrak{A}_1) = f_C(\mathfrak{A}_2)$. Therefore, we can treat logical terms as functions on universes (sets) rather than models, i.e., use $f_C(A)$ instead of $f_C(\mathfrak{A})$. I will do so in chapter 4, using C_A and $C_{\mathfrak{A}}$ as abbreviations.[19]

I can now give a semantic definition of (Tarskian) logical terms:

DEFINITION LT C is a (Tarskian) *logical term* iff C is a truth-functional connective or C satisfies conditions (A) to (E) above on logical constants.

I will call logical terms of the types (C.a) and (C.b) above *logical predicates* and *logical functors* respectively. Logical terms of type (C.c) I will call *logical quantifiers*, and logical terms of type (C.d) *logical quantifier functors*.

What kind of expressions satisfy (LT)? Clearly, all the logical constants of standard first-order logic do. Identity and the standard quantifiers are defined by total functions f_I, $f_\forall$, and $f_\exists$ on models such that, given a model $\mathfrak{A}$ with universe A,

(14) $f_I(\mathfrak{A}) = \{\langle a, b\rangle : a, b \in A \;\&\; a = b\}$,

(15) $f_\forall(\mathfrak{A}) = \{B : B = A\}$,

(16) $f_\exists(\mathfrak{A}) = \{B : B \subseteq A \;\&\; B \neq \varnothing\}$.

The definitions of the truth-functional connectives remain unchanged. Among the nonstandard terms satisfying (LT) are all Mostowskian quantifiers. As explained in chapter 2, these are n-place *predicative* quantifiers, i.e., quantifiers over n-tuples of predicates (where n is a positive integer, and a 1-tuple of predicates is a predicate). Among these are the following, redefined in the style of conditions (A) to (E).

(17) The 1-place "cardinal" quantifiers, defined, for any cardinal α by
$f_\alpha(\mathfrak{A}) = \{B : B \subseteq A \;\&\; |B| = \alpha\}$

(18) The 1-place quantifiers "finitely many" and "uncountably many," defined by
$f_{\text{finite}}(\mathfrak{A}) = \{B : B \subseteq A \;\&\; |B| < \aleph_0\}$
$f_{\text{uncountably many}}(\mathfrak{A}) = \{B : B \subseteq A \;\&\; |B| > \aleph_0\}$

(19) The 1-place quantifier "as many as not," defined by
$f_{\text{as many as not}}(\mathfrak{A}) = \{B : B \subseteq A \;\&\; |B| \geq |A - B|\}$

(20) The 1-place quantifier "most," defined by
$f_{M^1}(\mathfrak{A}) = \{B : B \subseteq A \;\&\; |B| > |A - B|\}$

(21) The 2-place quantifier "most," defined by
$f_{M^{1,1}}(\mathfrak{A}) = \{\langle B, C\rangle : B, C \subseteq A \;\&\; |B \cap C| > |B - C|\}$

We also have *relational* quantifiers satisfying (LT). One of these is,

(22) The "well-ordering" quantifier (a 1-place quantifier over 2-place relations), defined by $f_{WO}(\mathfrak{A}) = \{R : R \subseteq A^2 \;\&\; R$ is a strict linear ordering such that every nonempty subset of $\text{Fld}(R)$ has a minimal element in $R\}$.

I will call the logical terms below "relational quantifiers" as well:

(23) The second-level set-membership relation (a 2-place quantifier over pairs of a singular term and a predicate), defined by
$f_{\text{membership}}(\mathfrak{A}) = \{\langle a, B\rangle : a \in A \;\&\; B \subseteq A \;\&\; a \in B\}$

(24) The quantifier "ordering of the natural numbers with 0" (a 2-place quantifier over pairs of a 2-place relation and a singular term), defined by $f_{>_N,0}(\mathfrak{A}) = \{\langle R, a\rangle : R \subseteq A^2 \;\&\; a \in A \;\&$ and $\langle A, R, a\rangle$ is a structure of the natural numbers with their ordering relation and zero$\}$

Among *functors* and *quantifier functors* we have the following:

(25) The n-place "first" functors (over n-tuples of singular terms), defined, for any n, by $f_{\text{first}}(\mathfrak{A})$ = the function $g : A^n \to A$ such that for any n-tuple $\langle a_1, \ldots, a_n\rangle \in A^n$, $g(a_1, \ldots, a_n) = a_1$

(26) The 1-place "complement" quantifier functor (over 1-place predicates), defined by $f_{\text{complement}}(\mathfrak{A})$ = the function $g : \boldsymbol{P}(A) \to \boldsymbol{P}(A)$ such that for any $B \subseteq A$, $g(B) = A - B$

Examples of constants that do not satisfy (LT):

(27) The 1-place predicate "identical with **a**" (**a** is a singular term of the language), defined by $f_{=a}(\mathfrak{A}) = \{b : b \in A \;\&\; b = \mathbf{a}^{\mathfrak{A}}\}$, where $\mathbf{a}^{\mathfrak{A}}$ is the denotation of **a** in $\mathfrak{A}$

(28) The 1-place (predicative) quantifier "pebbles in the Red Sea," defined by $f_{\text{pebbles}\ldots}(\mathfrak{A}) = \{B : B \subseteq A \;\&\; B$ is a nonempty set of pebbles in the Red Sea$\}$

(29) The first-level membership relation (a 2-place first-level relation whose arguments are singular terms), defined by $f_{\in}(\mathfrak{A}) = \{\langle a, b\rangle : a, b \in A \;\&\; b$ is a set $\&\; a$ is a member of $b\}$

The definitions of these constants violate condition (E). To see why (29) fails, think of two models, $\mathfrak{A}$ and $\mathfrak{A}'$ with universes $\{0, \{0, 1\}\}$ and {Jean-Paul Sartre, Albert Camus} respectively. While the first-order structures $\langle\{0, \{0, 1\}\}, \langle 0, \{0, 1\}\rangle\rangle$ and $\langle$\{Jean-Paul Sartre, Albert Camus\}, $\langle$Jean-Paul Sartre, Albert Camus$\rangle\rangle$ are isomorphic (when taken as first-order, i.e., when the first elements are treated as sets of atomic objects), $\langle 0, \{0, 1\}\rangle \in f_{\in}(\mathfrak{A})$ but $\langle$Jean-Paul Sartre, Albert Camus$\rangle \notin f_{\in}(\mathfrak{A}')$.

Another term that is not logical under (LT) is the definite-description operator ι. If we define ι (a quantifier functor) by a function f that, given a model $\mathfrak{A}$ with a universe A, assigns to $\mathfrak{A}$ a partial function h from $\boldsymbol{P}(A)$ into A, then condition (C.d) is violated. If we make h universal, using some convention to define the value of h for subsets of A that are not singletons, it has to be shown that the convention does not violate (E). We can, however, construct a 2-place predicative logical quantifier "the," which expresses Russell's contextual definition of the description operator:

(30) $f_{\text{the}}(\mathfrak{A}) = \{\langle B, C\rangle : B \subseteq C \subseteq A \;\&\; B$ is a singleton set$\}$

7 A New Conception of Logic

The definition of logical terms in section 6 gives new meaning to "first-order logic" based on Tarski's ideas. "First-order logic" is now a schematic title for any system of logic with a complete collection of truth-functional connectives and a nonempty set of logical constants. It is open to us, the users, to choose which particular set of constants satisfying (LT) we want to include in our first-order system. The logic itself is an open framework: any term may be plugged in as a logical constant, provided this is done in accordance with conditions (A) to (E). Any first- or second-level *formal* term is acceptable, so long as it is incorporated into the system "in the right way." The general framework of logic based on this conception I will call *Unrestricted Logic* or UL. I will also refer to it as *Tarskian Logic*, since it is based on Tarski's conception of the task and structure of logic. A particular system of Tarskian logic is simply *a logic*. Both syntactically and semantically the new logic preserves the form of definition characteristic of standard mathematical logic: syntactically, the logical constants serve as "formula-building operators" on the basis of which the set of well-formed formulas is defined by induction; semantically, the logical constants are associated with pre-fixed rules, to be used in the recursive definition of satisfaction in a model. Thus, for example, the syntactic definition of the 2-place quantifier "most" is given by the following clause:

- If Φ and Ψ are well-formed formulas, then $(\text{Most}^{1,1} x)(\Phi, \Psi)$ is a well-formed formula.

The rule associated with "most" is expressed in the corresponding semantic clause:

- If Φ and Ψ are well-formed formulas, $\mathfrak{A}$ is a model with a universe A, and g is an assignment of individuals in A to the variables of the language, then

$\mathfrak{A} \models (\text{Most}^{1,1} x)(\Phi, \Psi)[g]$ iff

$\langle\{a \in A : \mathfrak{A} \models \Phi[g(x/a)]\}, \{a \in A : \mathfrak{A} \models \Psi[g(x/a)]\}\rangle \in f_{\mathbf{M}^{1,1}}(\mathfrak{A})$.

I will give a precise account of UL in chapter 4. In the meantime, I propose this provisional definition:

DEFINITION UL $\mathscr{L}$ is a logic in UL iff $\mathscr{L}$ is a Tarskian first-order system with (1) a complete set of truth-functional connectives and (2) a nonempty set of logical terms, other than those in (1), satisfying (LT).

I will now show (what should be clear from the foregoing discussion) that UL satisfies the pretheoretical requirements (C1) and (C2). Namely, if $\mathscr{L}$ is a first-order system in UL, then the Tarskian definition of "logical consequence" for $\mathscr{L}$ gives results in agreement with (C1) and (C2).

First the case for (C1). It suffices to show that the assumption (AS) (of section 2) holds for UL. Let $\mathscr{L}$ be any system of UL with new logical constants, let $\mathscr{C}$ be the logical vocabulary of $\mathscr{L}$, and let L be its extralogical vocabulary. The claim is that if Φ is a well-formed formula of $\mathscr{L}$, every possible extension of Φ relative to the vocabulary of $\mathscr{L}$ is represented by some model for $\mathscr{L}$ (where the extension of a sentence is taken to be a truth value, T or F).

I will sketch an outline of a proof. Suppose that Φ is an atomic formula of the form "Px," where P is an extralogical constant. The strong semantic variability of P and the other primitive terms in L ensures that every possible state of affairs relative to these terms is represented by some model $\mathfrak{A}$ for $\mathscr{L}$. So the claim holds for Φ. Now let Φ be of the form "$(Qx)\Psi x$," where Q is a quantifier and "Ψx" is (for the sake of simplicity) a formula with one free variable x. Assume the claim holds for "Ψx." Q, being a member of $\mathscr{C}$, is semantically rigid. Furthermore, its rigid interpretation is formal. But formal properties and relations intuitively do not change from one possible state of affairs to another. That is, while the number of, say, red things does vary among possible states of affairs, the second-level formal property "having n objects in X's extension" does not. Having n objects in a property's extension is always the same thing, no matter what property and what state of affairs we are considering. Therefore, the variability of situations with respect to "$(Qx)\Psi x$" is reduced to the variability of situations with respect to "Ψx." It is possible that "$(Qx)\Psi x$" has the extension T/F iff it is possible that "Ψx" has an extension representable by a subset B of the universe of some model $\mathfrak{A}$ such that $B \in f_Q(\mathfrak{A})/B \notin f_Q(\mathfrak{A})$. But by (the inductive) assumption, every possible extension of "Ψx" (relative to the vocabulary of $\mathscr{L}$) is represented by some model for $\mathscr{L}$. So if it is possible for "Ψx" to have an extension as required, there is a model that realizes this possibility. In this model the extension of "$(Qx)\Psi x$" is T/F. We can carry on this inductive reasoning with respect to any type of logical terms under (LT).

The case for (C2) is straightforward. Condition (E) expresses an intuitive notion of formality: to be formal is, intuitively, to take only structure into account. Within the scheme of model-theoretic semantics, to be formal is to be invariant under isomorphic structures. Now in UL, as in standard logic, logical consequences depend on the logical vocabulary of

the language. The formality of logical terms ensures that logical consequences do not rest on empirical evidence and do not distinguish the identity of objects in any given universe. Hence logical consequences of UL are formal in Tarski's sense.

Logics equivalent or similar to UL are often called in the literature "generalized logics," "extended logics," "abstract logics," or "model-theoretic logics." These labels may, however, convey the wrong message. Driving a wedge between "core" logic and its new "extensions," they seem to intimate that the "tight" and "lean" standard system is still the true logic. Such an interpretation of UL would, however, be wrongheaded. UL is not an abstract generalization of *real logic*. UL *is* real logic, full-fledged. As we have seen earlier in this chapter, the basic semantic principles of "core" logic (formulated by Tarski in the mid 1930s) are not fully materialized in the "standard" system. This system fails to produce all the formally necessary, i.e., "logical," consequences with a first-level vocabulary. It takes the full spectrum of UL logics to carry out the original program.

I have answered the question posed at the end of section 2. The broadest notion of logical term compatible with the intuitive concept of "logical consequence" is that of (LT). (LT) redefines the boundaries of logic, leading to the unrestricted system of UL. Condition (E) is especially important in determining the full scope of logic. It is worthwhile to trace the origins of this condition.

8 Invariance under Isomorphic Structures

The condition of invariance under isomorphic structures first appeared, as a characterization of logicality, in Lindenbaum and Tarski 1934–1935. Referring to a simple Russellian type-theoretic logic, Lindenbaum and Tarski proved a theorem that informally says, "Every relation between objects (individuals, classes, relations) which can be expressed by purely logical means [i.e., without using extralogical terms] is invariant with respect to every one-one mapping of the 'world' (i.e., the class of all individuals) onto itself."[20]

Now the metalanguage from which we draw the pool of logical terms is roughly equivalent to a subsystem of "pure" higher-order logic with Russellian simple types. For this language, Lindenbaum and Tarski's theorem shows that all definable notions satisfy the isomorphism condition with respect to "the world" (a "universal" model, in our terminology). The Lindenbaum-Tarski theorem appears to assume a notion of logicality that depends on the classification of the standard logical operators of a simple Russellian type theory as "purely logical." However, it follows from this very theorem that the standard operators themselves are invariant under isomorphic substructures, i.e., given any model $\mathfrak{A}$ (a submodel relative to Lindenbaum and Tarski's "universal" model) and a 1-place formula Φx, "$(\forall x)\Phi x$" is true in $\mathfrak{A}$ iff for any 1-place formula Ψx whose extension in $\mathfrak{A}$ is obtained from that of "Φx" by some permutation of the universe, "$(\forall x)\Psi x$" is true in $\mathfrak{A}$, and similarly for the other Russellian operators. So the theorem shows (relative to a simple type-theoretic language and the standard rules of logical proof) that Russellian logical terms and all terms that can be defined from them are "purely logical."

The idea that logical notions are distinguished by their invariance properties next appeared in Mautner's "An Extension of Klein's Erlanger Program: Logic as Invariant-Theory" (1946). Inspired by Klein's program of classifying geometrical notions in terms of invariance conditions, Mautner showed that standard mathematical logic can be construed as "invariant-theory of the symmetric group . . . of all permutations of the domain of individual variables."[21]

In his pioneering 1957 paper "On a Generalization of Quantifiers," Mostowski used the invariance property, for the first time, to license a genuine extension of standard first-order logic by adding new logical terms. Mostowski's condition technically was invariance under permutations of sets induced by permutations of the universe (of a given model). Informally, it was to be construed as the claim, (LQ2) of chapter 2, that quantifiers do not take into account the identity of individuals in the universe of discourse. Mostowski's criterion included references to the aforementioned papers of Lindenbaum and Tarski (1934 1935) and Mautner (1946).[22]

In 1966 Per Lindström generalized Mostowski's condition to full invariance under isomorphic (relational) structures, augmenting Mostowski's system with many-place predicative and relational quantifiers, often referred to as "Lindström quantifiers." There is a minor difference between Lindström's definition and (E) above: Lindström's structures are relational, and 0-place relations are not individuals but truth values, T or F. Thus mathematical structures involving individuals cannot be directly represented by logical terms, as in (24). Lindström, unlike Mostowski, was silent regarding the philosophical significance of his generalization. One might say his remarkable theorems solidify the distinguished status of standard first-order logic, but here again, it is unclear whether Lindström himself considers compactness and the Löwenheim-Skolem property to be essential ingredients of logicality or mere mathematically interesting features of one among many genuinely logical systems. This philosophical disengagement is characteristic of the abundant literature on "abstract logic" that has followed Lindström's work.[23]

I often wondered what Tarski would have thought about the conception of Tarskian logic proposed in this book. After the early versions of the present chapter had been completed, I came upon a 1966 lecture by Tarski, first published in 1986, that delighted me in its conclusion. In the lecture "What are Logical Notions?" Tarski proposed a definition of "logical term" that is coextensional with condition (E):

> Consider the class of *all* one-one transformations of the space, or universe of discourse, or "world" onto itself. What will be the science which deals with the notions invariant under this widest class of transformations? Here we will have . . . notions, all of a very general character. I suggest that they are the logical notions, that we call a notion "logical" if it is invariant under all possible one-one transformations of the world onto itself.[24]

The difference between Tarski's 1966 lecture and the earlier Lindenbaum and Tarski paper is that here Tarski explicitly talks about the scope of logical terms for a first-order framework. (Indeed, in his introduction to the posthumously published lecture, J. Corcoran suggests that we see it as a sequel to Tarski's 1936 "On the Concept of Logical Consequence," in which the scope of logical terms was left as an open question.) It follows from the above definition, Tarski now says, that no term designating an individual is a logical term; the truth-functional connectives, standard quantifiers, and identity are logical terms; Mostowski's cardinality quantifiers are logical, and in general, all predicates definable in standard higher-order logic are logical. Tarski emphasizes that according to his definition, any mathematical property can be seen as logical when construed as higher-order. Thus, as a science of individuals, mathematics is different from logic, but as a science of higher-order structures, mathematics is logic.

The analysis that led to the extension of "logical term" in Tarski's lecture is, however, different from that proposed here. Tarski, like Mautner, introduced his conception as a generalization of Klein's classification of geometrical disciplines according to the transformations of space under which the geometrical concepts are invariant. Abstracting from Klein, Tarski characterized logic as the science of all notions invariant under one-to-one transformations of the universe of discourse ("space" in a generalized sense). My own conclusions, on the other hand, are based on analysis of Tarski's early work on the philosophical foundations of logic.

This is the reason that, unlike in the later Tarski, the criterion for logical terms proposed here includes, but is not exhausted by, condition (E). To be a logical term is not just to be a higher-level, mathematical term; it is to be incorporated in a certain syntactic-semantic system in a way that allows us to identify all intuitively logical consequences by means of a given rule, e.g., Tarski's (LC).

Following Lindström (Tarski's 1966 lecture remained unknown for a long time), condition (E) has been treated by mathematical logicians as a criterion for *abstract* logical terms. In the last decade condition (E), and some variants thereof, began to appear as a criterion of logicality in the formal semantic literature, often in combination with other criteria, like conservativity. If my analysis is correct, conservativity and other linguistic properties constraining (E) have nothing to do with logicality.

The only thorough philosophical discussion of condition (E) that I know of appears in Timothy McCarthy's 1981 paper "The Idea of a Logical Constant."[25] McCarthy rejects (E) as a sufficient condition for "logicality" on the grounds that it does not prevent the definition of logical terms by means of "contingent" expressions. To illustrate McCarthy's point, let us consider the quantifier "the number of planets," defined by

$$f_{\text{the number of planets}}(\mathfrak{A}) = \{B : B \subseteq A \ \& \ |B| = \text{the number of planets}\}.$$

Clearly, the quantifier "the number of planets" satisfies (E). Now

(31) The number of planets = 9

is contingent in the metalanguage, i.e, its extension changes from one "possible world" (in which we interpret the metalanguage) to another. Consider the sentence

(32) (The number of planets x)(Px & $\sim Px$).

This sentence is logically false as a matter of fact, McCarthy would say, that is, as a matter of the fact that the number of planets is larger than zero. However, in the counterfactual situation in which our sun had no satellites, (32) would turn out logically true. Therefore, "the number of planets x" will not do as a logical quantifier.

McCarthy's objection, however, does not affect my criterion, which includes conditions (A) to (D) in addition to (E). Condition (B) states that logical terms are identified with their (actual) extensions, so that in the metatheory the definitions of logical terms are rigid. Qua quantifiers, "the number of planets" and "9" are indistinguishable. Their (actual) extensions determine one and the same formal function over models, and this

function is a legitimate logical operator. In another world another description (and possibly another symbol) may designate this function. But that has no bearing on the issue in question. Inscription (32) may stand for different statements in different worlds. But the logical statement (32) is the same, and false, in all worlds. For that reason logic—Unrestricted Logic or any logic—is invariant across worlds. From the point of view of logic presented here, McCarthy's demand that the meaning of logical terms be known a priori is impertinent. The question is not how we come to know the meaning of a given linguistic expression, but how we set out to use it. If we set it up as a rigid designator of some formal property in accordance with conditions (A) to (E), it will work well as a logical constant in any Tarskian system of logic. Set differently, it might not. Switching perspectives, we may say that the only way to understand the meaning of a term used as a logical constant is to read it rigidly and formally, i.e., to identify it with the mathematical function that semantically defines it.

9 Conclusion

We have arrived at a general theory of the scope and nature of logical terms based on analysis of the function of logic and the philosophical guidelines at the basis of modern semantics. Given the breadth of the logical enterprise, we discovered that the standard terms alone do not provide an adequate superstructure. Yet in view of its goal, not every term can be used as a constant in Tarskian logic. There exists a clear, unequivocal criterion for eligible terms, and the terms satisfying this criterion far exceed those of "standard" logic.

We can now answer the questions posed at the end of chapter 2. Mostowski's claim that standard mathematical logic does not exhaust the scope of first-order logic has been vindicated. His semantic criterion on quantifiers, namely, "not distinguishing the identity of individuals in the universe," is most naturally interpreted as not discerning the difference between isomorphic structures. As for logicality and cardinality, the invariance condition implies that the two coincide in the case of predicative quantifiers, but in general, these notions are not essentially connected.

The next task is to outline a complete system of first-order logic with logical terms satisfying (LT). The series of conditions proposed in the present chapter constitute a definition of logical terms "from above": one can understand the conditions without thereby knowing how to construct all constants possessing the required properties. In the next chapter I will give a *constructive* definition of logical constants, inspired by Mostowski.

Mostowski's correlation of quantifiers with cardinality functions did to "predicative" generalized logic what the association of connectives with Boolean truth functions earlier did to sentential logic. It provided a highly informative answer to the questions, "What is a predicative quantifier?" "What are all the predicative quantifiers?" Following Mostowski, I will present a correlation of logical terms with mathematical functions of a certain kind so that the totality of functions will determine the totality of logical terms and each function will embed the "instructions" for constructing one logical term from the total list.

Chapter 4
Semantics from the Ground Up

Our philosophical analysis in the last chapter has led to the conclusion that any second-level mathematical predicate can be construed as a logical quantifier under a semantic definition satisfying the metatheoretical conditions (A) to (E). Since the predicative quantifiers defined in chapter 2 satisfy these conditions, they are genuine logical quantifiers, and Mostowski's claim that they belong in a systematic presentation of symbolic logic is justified. Our analysis also provides an answer to the question "Which second-level predicates on relations are logical quantifiers?" Relational quantifiers are simply logical terms of a particular type: second-level predicates or relations whose arguments include at least one first-level relation (many-place predicate).

On my analysis, Mostowski's semantic condition on predicative quantifiers, (LQ2), the requirement that quantifiers should not distinguish the identity of elements in the universe of a given model, corresponds to Tarski's (C2), the requirement that logical terms (and hence logical quantifiers) be *formal*. Like Mostowski, I interpret (C2) as an invariance condition, and this condition, when applied to predicative quantifiers, coincides with his. More accurately, Mostowski's rendering of (LQ2) as invariance under permutations of sets induced by permutations of the universe is generalized to condition (E), which says that logical terms in general are invariant under isomorphic structures. In terms of Mostowski's definition of quantifiers as functions from sets to truth values, I say that a logical term over universe A is a function q from sequences of relations (predicates, individuals) of the right type to truth values, T or F, such that if s is a sequence in $\mathrm{Dom}(q)$ and m is a permutation of A,

$$q(s) = \mathrm{T} \text{ iff } q(m(s)) = T,$$

where $m(s)$ is the image of s under m.

The characterization of logical constants in terms of invariance under permutations of the universe is still not very informative, however. In the case of predicative quantifiers, Mostowski was able to establish a one-to-one correspondence between quantifiers satisfying (LQ2) and cardinality functions of a specified kind, and this resulted in a highly informative characterization of predicative quantifiers: predicative quantifiers attribute *cardinality* properties (relative to the cardinality of a given universe) to the extensions of 1-place first-level predicates in their scope; the functions t associated with predicative quantifiers constitute "rules" for constructing predicative quantifiers over a universe A. Although cardinality functions can be extended to logical terms other than predicative quantifiers, they evidently will not cover all the logical terms over a universe A. The latter express *structural* properties of sets, relations, and individuals in general, not just cardinality properties.

My main goal in the present chapter is to develop a semantic definition of logical terms that captures the idea of *formal structure* in a way analogous to that in which Mostowski's definition captures the idea of cardinality. Mostowski's definition distinguishes sets according to their size relative to the size of a given universe. I want to characterize all formal patterns of individuals standing in relations within an arbitrary universe A and then distinguish relations according to the formal patterns they exhibit. This will be the basis for my "constructive" definition of logical terms over A. But first I will examine the original characterization of logical terms satisfying (E), due to Per Lindström.

1 Lindström's Definition of "Generalized Quantifiers"

In "First Order Predicate Logic with Generalized Quantifiers" Lindström (1966a) associates generalized quantifiers with classes of structures (models) closed under isomorphism. More precisely, his semantic definition goes as follows:

DEFINITION LQ A quantifier is (semantically) a class $\bar{\mathrm{Q}}$ of relational structures of a single type $t \in \omega^n$, $n > 0$, closed under isomorphism,

where a relational structure is a sequence consisting of a universe (a set) and a series of constant relations on, or subsets of, the universe (but not individuals). The type of structure $\mathfrak{A}$ is an ordered n-tuple, $\langle m_1, \ldots, m_n \rangle$, where n is the number of constant relations R_i in $\mathfrak{A}$ and m_i, $1 \leq i \leq n$, is the number of arguments of the relation R_i. (A truth value is considered by Lindström a relation with no arguments. There are only two 0-place relations, T and F.) Each semantic quantifier $\bar{\mathrm{Q}}$ is symbolized by a syntactic quantifier Q; different syntactic quantifiers corresponding to different semantic quantifiers. If Q symbolizes $\bar{\mathrm{Q}}$, Q is said to be of the type t common to all the structures in $\bar{\mathrm{Q}}$. A syntactic quantifier Q of type $t = \langle m_1, \ldots, m_n \rangle$ is a quantifier in $m_1 + m_2 + \cdots + m_n$ variables that attaches to n formulas to form a new formula.

The truth conditions for formulas with Lindström quantifiers are defined as follows: Let Q be a Lindström quantifier of type $t = \langle m_1, \ldots, m_n \rangle$. Let $\Phi_1, \ldots, \Phi_n$ be formulas of first-order logic with Lindström quantifiers. Let $\bar{x}_1, \ldots, \bar{x}_n$ be a series of n pairwise disjoint elements, where for $1 \leq i \leq n$, the element $\bar{x}_i$ is a series of m_i distinct variables. Let $\mathfrak{A}$ be a model with universe A, and let g be an assignment of elements in A to the individual variables of the language. Then

$\mathfrak{A} \models (\mathrm{Q}\bar{x}_1, \ldots, \bar{x}_n)(\Phi_1, \ldots, \Phi_n)[g]$ iff the structure $\langle A, \Phi_1^{\mathfrak{A}}\bar{x}_1[g], \ldots, \Phi_n^{\mathfrak{A}}\bar{x}_n[g] \rangle$ is a member of $\bar{\mathrm{Q}}$,

where for $1 \leq i \leq n$,

$$\Phi_i^{\mathfrak{A}}\bar{x}_i[g] = \begin{cases} \mathrm{T} & \text{if } \bar{x}_i = \langle\ \rangle \text{ and } \mathfrak{A} \models \Phi_i[g] \\ \mathrm{F} & \text{if } \bar{x}_i = \langle\ \rangle \text{ and } \mathfrak{A} \not\models \Phi_i[g] \\ \{\bar{a}_i : \mathfrak{A} \models \Phi_i[g(\bar{x}_i/\bar{a}_i)]\} & \text{otherwise} \end{cases}$$

("$\bar{a}_i$" stands for an arbitrary sequence of m_i elements of A, $a_{i_1}, \ldots, a_{i_{m_i}}$, and "$g(\bar{x}_i/\bar{a}_i)$" abbreviates "$g(x_{i_1}/a_{i_1}) \ldots (x_{i_{m_i}}/a_{i_{m_i}})$").

Clearly, the quantifiers definable in Lindström's logic include all the logical quantifiers of chapter 3 over (sequences of) predicates and relations (but not over sequences including individuals). In addition, all the logical predicates and all the truth-functional connectives are definable as Lindström quantifiers. Thus we have the following:

(1) The existential quantifier of standard logic is defined as $\bar{\mathrm{E}}$ = the class of all structures $\langle A, P \rangle$, where A is a set, $P \subseteq A$, and P is not empty.

(2) The predicative quantifier R^2 of chapter 2 ("there are more . . . than - - -") is defined as $\bar{R}^2$ = the class of all structures $\langle A, P_1, P_2 \rangle$, where A is a set, $P_1, P_2 \subseteq A$, and $|P_1| > |P_2|$.

(3) The "well-ordering" relational quantifier of chapter 3, WO, is defined as $\overline{\mathrm{WO}}$ = the class of all structures $\langle A, R \rangle$, where A is a set, $R \subseteq A^2$, and R well-orders Fld(R).

(4) The negation of sentential logic is defined as $\bar{\mathrm{N}}$ = the class of all structures $\langle A, F \rangle$, where A is a set. (The structure $\langle A, F \rangle$ is non-isomorphic to $\langle A, T \rangle$ by definition.)

(5) The disjunction of sentential logic is defined as $\dot{D}$ = the class of all structures $\langle A, S_1, S_2\rangle$, where A is a set and S_1, S_2 are truth values, at least one of which is T.

My definition of logical terms in chapter 3 essentially coincides with Lindström's. There are some small differences in the construction of models: Lindström's models include the two truth values T and F as components. This allows him to construe the truth-functional connectives as logical quantifiers. (Indeed, I could incorporate the same device in my theory.) In addition, Lindström does not consider structures with individuals. It is easy, however, to extend his definition to structures of this kind, and given such an extension, all logical terms of (LT) will fall under Lindström's definition. There is also a minor difference between Lindström's syntax and mine: whereas I constructed an n-place predicative quantifier as binding a single individual variable in any n-tuple of well-formed formulas in its domain, Lindström's predicative quantifiers bind n distinct variables. Thus what I symbolize as

$$(Qx)(\Phi_1 x, \ldots, \Phi_n x)$$

Lindström symbolizes as

$$(Qx_1, \ldots, x_n)(\Phi_1 x_1, \ldots, \Phi_n x_n).$$

However, since the two quantifications express exactly the same statement, the difference just amounts to a simplification of the notation.

In chapter 1, I pointed out that the apparatus of Tarskian model-theoretic semantics is "too rich" for standard first-order logic. We never use the model-theoretic apparatus in its entirety to state the truth conditions of sentences of standard logic, to determine standard logical truths and consequences, to distinguish semantically between nonequivalent standard theories, etc. In particular, the collection of infinite models is to a large extent redundant because any sentence or theory represented by an infinite model is represented by uncountably many distinct infinite models (the Löwenheim-Skolem-Tarski theorem). The new conception of logic, which received its first full-scale expression in Lindström, enriches the expressive power of the first-order language so that the model-theoretic apparatus is put to full use. The extended logical vocabulary allows the formation of new sentences and theories, so evey model becomes the unique representation (up to isomorphism) of some theory of the new language. Put otherwise, every structure, up to isomorphism, is describable by a theory of the generalized language, indeed, in Lindström's system, by a single sentence (if the language has enough nonlogical constants of the "right" type). Thus, let $\mathfrak{A} = \langle A, R_1, \ldots, R_n\rangle$ be a structure of type $t = \langle m_1, \ldots, m_n\rangle$. Let $\bar{Q}$ be the class of all structures $\mathfrak{B}$ isomorphic to $\mathfrak{A}$, and let Q be the quantifier defined by $\bar{Q}$. Let $P_1, \ldots, P_n$ be distinct relational constants of $m_1, \ldots, m_n$ places respectively (P_i being a sentential letter if $m_i = 0$), and let $\bar{x}_1, \ldots, \bar{x}_n$ be series of distinct variables as explained above. Then the sentence

(6) $(Q\bar{x}_1, \ldots, \bar{x}_n)(P_1\bar{x}_1, \ldots, P_n\bar{x}_n)$

describes the unique structure $\mathfrak{A}$ (up to isomorphism).

Lindström's definition, however, is "from above." As such, it does not show us how to "construct" logical terms over a model $\mathfrak{A}$ using elements in the universe of $\mathfrak{A}$ as the initial building blocks. In addition, Lindström's definition of logical terms over a specific model $\mathfrak{A}$ involves quantification (in the metalanguage) over all models. Thus, to determine whether an n-tuple of formulas $\langle\Phi_1, \ldots, \Phi_n\rangle$ satisfies a quantifier Q in $\mathfrak{A}$, we need information not only on the extentions of $\Phi_1, \ldots, \Phi_n$ in $\mathfrak{A}$ but also about the class of all models for the language. In the next section I will propose a definition of logical terms "from the ground up." This definition shows how to build logical terms over $\mathfrak{A}$ out of constructs of elements of $\mathfrak{A}$ and without reference to the totality of models.

2 Constructive Definition of Logical Terms

The idea is this: Tarskian logical terms over a model $\mathfrak{A}$ with universe A distinguish the *form* or *structure* of sets, relations, and functions over A. Any two relations differing in structure will be distinguished by a logical term on A, but relations that share the same structure will not. Similarly for sets and functions. So, to define the totality of logical terms on A, we first have to define the totality of "structures" over A. Once we determine the totality of, say, structures of binary first-level relations over A, we can define 1-place binary relational quantifiers on A as functions that assign the value T to some of these structures but not to others (allowing, of course, for the two extreme cases of functions that assign the value T to all binary relational structures, and to none). The totality of these functions is the totality of binary relational quantifiers on A. The definition will be general enough to include all types of logical terms. For the sake of simplicity I will, however, omit logical functors and logical quantifier functors. It is easy to extend the definition to these logical terms as well.

Before I begin the formal presentation, I will explain the idea behind the definition by reference to a simple example.

An informal account

Suppose we have a universe with ten individuals, say Alan, Becky, Carl, Debra, Eddy, Fred, Gary, Helen, Ian, and Jane. We want to identify all structures involving these persons that are the extensions of (legitimate) first-order logical terms over a model $\mathfrak{A}$ with the above group as its universe. I will refer to this universe simply as "The Group."

Let us consider several structures involving members of the Group (designated by their initials):

(7) j

(8) $\{a, c, d, i\}$

(9) $\{a, b, c, d, e, f, g, h, i, j\}$

(10) $\{\{a, c, d, i\}\}$

(11) $\{\{a, b, c, d, e, f, g, h, i, j\}\}$

(12) $\{\{a\}, \{c\}, \{d\}, \{h\}\}$

(13) $\{\{a\}, \{b\}, \{c\}, \{d\}, \{e\}, \{f\}, \{g\}, \{h\}, \{i\}, \{j\}\}$

(14) $\{\langle a, a\rangle, \langle f, f\rangle, \langle g, g\rangle, \langle j, j\rangle\}$

(15) $\{\langle a, a\rangle, \langle b, b\rangle, \langle c, c\rangle, \langle d, d\rangle, \langle e, e\rangle, \langle f, f\rangle, \langle g, g\rangle, \langle h, h\rangle,$
$\langle i, i\rangle, \langle j, j\rangle\}$

(16) $\{\varnothing, \{\langle a, j\rangle\}, \{\langle a, j\rangle, \langle c, d\rangle, \langle i, h\rangle\}, \{\langle a, j\rangle, \langle c, h\rangle, \langle g, d\rangle\}\}$

(17)
$$\begin{array}{lll}
\{\{\langle a, b\rangle, \langle b, c\rangle, \langle a, c\rangle\}, & \cdots & \{\langle j, a\rangle, \langle a, b\rangle, \langle j, b\rangle\}, \\
\{\langle a, b\rangle, \langle b, d\rangle, \langle a, d\rangle\}, & \cdots & \{\langle j, a\rangle, \langle a, c\rangle, \langle j, c\rangle\}, \\
\{\langle a, b\rangle, \langle b, e\rangle, \langle a, e\rangle\}, & \cdots & \{\langle j, a\rangle, \langle a, d\rangle, \langle j, d\rangle\}, \\
\vdots & \vdots & \vdots \\
\{\langle a, j\rangle, \langle j, g\rangle, \langle a, g\rangle\}, & \cdots & \{\langle j, i\rangle, \langle i, f\rangle, \langle j, f\rangle\}, \\
\{\langle a, j\rangle, \langle j, h\rangle, \langle a, h\rangle\}, & \cdots & \{\langle j, i\rangle, \langle i, g\rangle, \langle j, g\rangle\}, \\
\{\langle a, j\rangle, \langle j, i\rangle, \langle a, i\rangle\}, & \cdots & \{\langle j, i\rangle, \langle i, h\rangle, \langle j, h\rangle\}\}
\end{array}$$

(18) $\{\varnothing, \langle\{\langle a, j\rangle\}, b\rangle, \langle\{\langle c, d\rangle, \langle i, h\rangle\}, e\rangle,$
$\langle\{\langle c, h\rangle, \langle g, d\rangle\}, f\rangle\}$

(19)
$$\begin{array}{lll}
\{\langle\{\langle a, b\rangle, \langle b, c\rangle, \langle a, c\rangle\}, a\rangle, & \cdots & \langle\{\langle j, a\rangle, \langle a, b\rangle, \langle j, b\rangle\}, j\rangle, \\
\langle\{\langle a, b\rangle, \langle b, d\rangle, \langle a, d\rangle\}, a\rangle, & \cdots & \langle\{\langle j, a\rangle, \langle a, c\rangle, \langle j, c\rangle\}, j\rangle, \\
\langle\{\langle a, b\rangle, \langle b, e\rangle, \langle a, e\rangle\}, a\rangle, & \cdots & \langle\{\langle j, a\rangle, \langle a, d\rangle, \langle j, d\rangle\}, j\rangle, \\
\vdots & \vdots & \vdots \\
\langle\{\langle a, j\rangle, \langle j, g\rangle, \langle a, g\rangle\}, a\rangle, & \cdots & \langle\{\langle j, i\rangle, \langle i, f\rangle, \langle j, f\rangle\}, j\rangle, \\
\langle\{\langle a, j\rangle, \langle j, h\rangle, \langle a, h\rangle\}, a\rangle, & \cdots & \langle\{\langle j, i\rangle, \langle i, g\rangle, \langle j, g\rangle\}, j\rangle, \\
\langle\{\langle a, j\rangle, \langle j, i\rangle, \langle a, i\rangle\}, a\rangle, & \cdots & \langle\{\langle j, i\rangle, \langle i, h\rangle, \langle j, h\rangle\}, j\rangle\}
\end{array}$$

How shall we decide which of these structures are the extensions of logical terms over a model $\mathfrak{A}$ with the Group as its universe? The answer follows directly from the criterion for logical terms in chapter 3: a structure is the extension of a legitimate logical term iff it is closed under permutations of the universe. I will call such a structure a *logical structure*. Thus if S is a logical structure that contains the element E, then S also contains every element E' that can be obtained from E by some permutation of the universe. Let us examine each of the above structures and see what kind of structure it is.

Structure (7) consists of a particular member of the Group, Jane. Jane is not preserved under permutations of the Group, because such permutations may assign Fred to Jane, and Fred is not Jane. Jane (like Fred, Ian, and the rest) is not a "logical individual." Indeed, it is a basic principle of logic that there are no logical individuals and individuals do not constitute the extension of any logical term.

Structure (8) is also not closed under permutations of the universe. A permutation that assigns Jane to Alan, Alan to Carl, Helen to Debra and Gary to Ian, will carry us beyond $\{a, c, d, i\}$ to $\{a, g, h, j\}$. Here (8) may be the extension of the first-level predicate "x is redheaded," or "x is a leftist." But (8) does not represent any first-level logical property of members of the Group.

Structure (9), on the other hand, does represent a first-level logical property, since (9) is preserved under all permutations of the universe. Thus no matter who is assigned to Jane by a given permutation m, this person is already in (9). Put differently, the universal set is its own image under all permutations of the universe. We can associate with this set the property of being a member of the Group or see it as the property of being American, etc. No matter what other properties are "extentiated" in the Group by the universal set, (9) is also an instantiation of the logical property of self-identity over the Group and hence is a logical structure.

Structure (10), like (8), is not logical. It may be the extension of the second-level predicate "P is a property of redheads," or "P is an attribute of leftists." But these do not coincide with any second-level logical properties of members of the Group.

Structure (11), however, is the extension of a logical term, namely the universal quantifier over the Group.

Structure (12) is also nonlogical, since it is not closed under permutations of the universe. Suppose that among the members of the Group Alan is the only philosopher, Helen is the only linguist, Carl is the only historian, and Debra is the only novelist. Then (12) may be the extension of the nonlogical second-level predicate "P is either a distinctive characteristic of philosophers, a distinctive characteristic of linguists, a distinctive characteristic of historians, or a distinctive characteristic of novelists." But (12) cannot be the extension of any logical term over the Group.

Structure (13), unlike (12), is logical. Structure (13) is the extension of the quantifier "there is exactly one x such that" over $\mathfrak{A}$. As a predicate, (13) is the second-level attribute "P is a property of exactly one individual," an attribute whose extension is invariant under permutations of the Group.

Structure (14) too is nonlogical. Structure (14) may be the extension of "x likes y's dog(s)" over the Group (each dog owner likes his own dog(s)), or it may be the extension of some other relation over the Group, but the relation in question is not logical, and (14) cannot exhaust the extension of any logical term over the Group.

Structure (15) is the familiar relation of identity. This relation is closed under permutation of the universe and hence is logical.

Structure (16) may be the extension of the second-level predicate "X is the set of married pairs (husband and wife) in 1981, or X is the set of married pairs in 1982, or . . . , or X is the set of married pairs in 1990." Thus (16) reflects the various matrimonial constellations within the Group in the last decade. For example, during the first five years there were no marriages among members of the Group. Then in 1986 Alan married Jane, in 1987 Carl married Debra and Ian married Helen, and in 1989 Debra divorced Carl and married Gary, while Carl married Helen, who divorced Ian. This chronicle is clearly not closed under permutations of members of the Group.

Structure (17), on the other hand, is closed under permutations. It represents a linear ordering of triples in general. Structure (17) makes up the extension of the relational quantifier "R is a strict linear ordering of triples." This quantifier, symbolized by Q, will appear in formulas of the form "$(\mathrm{Q}xy)\Phi$." Thus if three members of the Group graduated from Columbia College, and their graduation dates do not coincide, the statement "$(\mathrm{Q}xy)$ x graduated from Columbia College before y" will turn out true in the universe in question.

Another nonlogical structure is given by (18). Suppose that there are three children in the Group: Becky, born to Alan and Jane in 1986, Eddy, born to Carl and Debra in 1987, and Fred, born to Gary and Debra in 1989. A second-level predicate that records births in the Group next to weddings (of men to women, by year, as in (16)), may have (18) as its extension.

Finally, (19) is a logical structure of pairs consisting of a strict linear ordering of a triple and its smallest element. This structure "extentiates" a relational quantifier over pairs of a binary relation and an individual, similar to (24) of chapter 3.

The principle of closure under permutations determines all the logical terms over a given universe. Every structure containing sets of individuals, relations of individuals, sequences of these, or sequences of sets/relations and individuals and closed under permutations of the universe determines a legitimate logical term over that universe. But the principle of closure under permutations can be used not only to identify but also to construct logical structures over a universe A. The construction of such structures is a very simple matter.

Again, take the Group. Construct any set of members of the Group, say $\{a, b, d, f\}$. Examine all permutations m of members of the Group and for each such permutation m add $m(a)$, $m(b)$, $m(d)$ and $m(f)$ to your set. In other words, close the set $\{a, b, d, f\}$ under all permutations of the universe, or create a union of all its images under such permutations. You will end up extending $\{a, b, d, f\}$ to (9), the universal set of the domain. This set is the extension of the first-level logical predicate of self-identity over the Group.

In a similar manner you can start from the relation (14), and by uniting (creating a union of) all its images under permutations of the universe, you will obtain the logical structure (15), the extension of the binary logical relation of identity.

Likewise, (17) can be obtained from $\{\{\langle a, b\rangle, \langle b, j\rangle, \langle a, j\rangle\}\}$ by closing it under permutations. And so on.

Suppose now you start with $\{\varnothing, \{a\}, \{a, b\}\}$. Closure under permutations will give you a set whose members are the empty set, all unit sets, and all sets of two elements. This set is the extension of the 1-place predicative quantifier "there are at most two" over the Group.

I have characterized the logical terms over a single universe, but my theory of logical terms says that logical terms do not distinguish between universes of the same cardinality. That is, each logical term is defined by a rule that does not change from one universe of cardinality α to another. Thus, although the characterization of identity for the Group by (15) would do, this is evidently not an adequate characterization for all universes with 10 elements. To capture the idea of a logical term, the *rule* associated with such a term, rather than its extension in a particular universe, should be specified. A very simple method of associating terms with rules presents itself. The idea is this: instead of recording the actual extension of a given term in a given universe, let us record its "index extension." Unlike its "object extension," the index extension encodes a rule that applies to all universes of the same cardinality. We can then distinguish between rules that do, and rules that do not, correspond to logical terms over universes of the cardinality in question.

I will begin by specifying a fixed index set for all universes of a given cardinality. In case of the Group, I will take 10, identified with the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$, as my index set. More generally, if A is a universe of cardinality α, I will take the least ordinal of cardinality α, defined as the set of all smaller ordinals, to be a standard index set for all universes of cardinality α. I will say that *A is indexed by α* or, in the example above, that the Group is indexed by 10. There are, of course, many ways of indexing the Group by 10. We may start any way we want, say assigning 0 to Alan, 1 to Becky, and so on, following the alphabetical order of the members's first names. Next we associate with each structure generated from members of the Group its index image under the chosen indexing. Thus the index image of (14) is

(20) $\{\langle 0, 0\rangle, \langle 5, 5\rangle, \langle 6, 6\rangle, \langle 9, 9\rangle\}$.

The index image of (15) is

(21) $\{\langle 0, 0\rangle, \langle 1, 1\rangle, \langle 2, 2\rangle, \langle 3, 3\rangle, \langle 4, 4\rangle, \langle 5, 5\rangle, \langle 6, 6\rangle, \langle 7, 7\rangle, \langle 8, 8\rangle, \langle 9, 9\rangle\}$.

And the index images of (7), (9), (11), and (16) are respectively

(22) 9,

(23) $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$,

(24) $\{\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}\}$,

(25) $\{\varnothing, \{\langle 0, 9\rangle\}, \{\langle 0, 9\rangle, \langle 2, 3\rangle, \langle 8, 7\rangle\}, \{\langle 0, 9\rangle, \langle 2, 7\rangle, \langle 6, 3\rangle\}\}$.

Note that it is essential that we do not treat the members of 10 in the same way that we treat 10, namely as sets of all smaller numbers. The reason is that if we identify 9 with $\{0, 1, 2, \ldots, 8\}$, (22) will represent not only (7) but also

(26) $\{a, b, c, d, e, f, g, h, i\}$.

Similarly, if we identify 0 with $\varnothing$, (25) will not distinguish between (16) and

(27) $\{a, \{\langle a, j\rangle\}, \{\langle a, j\rangle, \langle c, d\rangle, \langle i, h\rangle\}, \{\langle a, j\rangle, \langle c, h\rangle, \langle g, d\rangle\}\}$.

Therefore, I define an *index set* to be a set of ordinals treated as individuals (or as sets of pairs of the form (β, a), where a is some fixed object). More precisely, an index set for a universe of cardinality α is the set of all ordinals smaller than the least ordinal of cardinality α, where the ordinals in the index set are themselves not sets of ordinals.

Back to the index set 10. I call a member of 10 a *10-individual*, a subset of 10 a *10-predicate*, and a set of n-tuples of members of 10 ($n > 1$) a *10-relation*. Thus (22) is a 10-individual, (23) is a 10-predicate, and (20) and (21) are 10-relations.

I call any finite sequence of 10-individuals, 10-predicates, and/or 10-relations a *10-argument*. Such sequences constitute the arguments of logical terms over the Group. It follows that a 10-individual is a *10-predicate-argument*; a finite sequence of two or more 10-individuals is a *10-relation-argument*; other 10-arguments are *10-quantifier-arguments*. I say that 10-arguments are of the same type if they have the same structure: all individuals are of the same type, all sets of individuals are of the same type, and all n-place relations of individuals are of the same type. Sequences of m elements of corresponding types are also of the same type. (The formal definition of type is slightly different, but the notion of "same type" is the same.) Thus

(28) $\langle 1, 2\rangle$

and

(29) $\langle 3, 4\rangle$

are of the same type, and so are

(30) $\{8\}$

and

(31) $\{3, 4, 5, 8\}$,

as well as

(32) $\langle 1, \{1, 2\}, \{\langle 1, 3\rangle\}\rangle$

and

(33) $\langle 9, \{3, 4, 5\}, \{\langle 6, 7\rangle, \langle 7, 6\rangle\}\rangle$.

I call two 10-arguments *similar* iff one is the image of the other under some permutation of 10. Thus (28) and (29) are similar, but neither (30) and (31) nor (32) and (33) are. Looking at the logical structures among (7) through (19), we see that a logical structure is a structure of similar elements of a given type. More accurately, a logical structure over the Group is a structure of 10-arguments of a single type closed under the relation of similarity. Since the relation of similarity is an equivalence relation, each logical structure corresponds to a union of equivalence classes of similar 10-arguments of a given type.

Note that while some logical terms can be identified with a single equivalence class, others correspond to a union of equivalence classes. For example, "there is *exactly* one" is a function that gives to a 10-argument the value T iff it is a member of the equivalence class of all sets similar to $\{0\}$, $[\{0\}]$, but "there is *at least* one" assigns the value T to members of more than one equivalence class. So I define a logical term over universes with 10 elements as a function from all equivalence classes of a given type to $\{T, F\}$. "There is exactly one" assigns the value T to the equivalence class $[\{0\}]$ and F to all other equivalence classes of subsets of the universe, whereas "there is at least one" assigns the value T to $[\{0\}]$, $[\{0, 1\}]$, ..., $[\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}]$ and F to $[\varnothing]$. I call such functions *10-operators*.

What can we do with 10-operators? Well, there are several things we can do. We can take a 10-operator of type t (that is, an operator defined over equivalence classes of elements of type t), a structure of the same type generated from the Group (a 10-individual being matched with a member of the Group, etc.), and ask whether the latter satisfies the logical term defined by the former. For example, we can take the extension of the predicate "x is a philosopher," namely $\{a\}$, and ask whether it satisfies a given 1-place predicative quantifier over the Group. To find the answer, we first index the Group by 10 (in any way we choose). Then we take the index image of $\{a\}$ and see whether the quantifier in question (defined as a 10-operator) gives the value T to $[\{\text{index}(a)\}]$. This test will show that

(34) (At least one x) x is a philosopher

is true in the intended model of the Group (Alan is a philosopher), but

(35) $(!2\,x)$ x is a philosopher

and

(36) $(\forall x)$ x is a philosopher

are false in the same model (Alan is the only philosopher).

Second, we can take a structure over the Group, ask whether it defines a logical term over the Group, and, if the answer is positive, get a general semantic schema of the logical term in question. We do this by creating an index image of the structure and examining whether the result is a union of equivalence classes under the relation of similarity. Thus (21), an index image of (15), is an equivalence class of all pairs similar to $\langle 0, 0\rangle$ and therefore (15) does determine a logical term, namely identity, over the

Group. The index image (20) of (14) does not constitute such an equivalence class (or a union of equivalence classes under similarity), and hence (14) does not determine a logical term over the Group.

Third, we can take any 10-operator and use it as a blueprint for constructing a logical term over the Group. Thus, starting with any indexing of the Group by 10, I take the 10-operator "exactly one," a function o from all equivalence classes of subsets of 10 to $\{T, F\}$ defined as

$$o[N] = T \text{ iff } [N] = [\{0\}],$$

and transform it into a quantifier in extension by going through the elements of the equivalence class(es) assigned T and constructing their correlates over the Group: $\{a\}$, $\{b\}$, etc. I then collect these correlates into a set, and this is (13), the quantifier "there is exactly one" over the Group.

Finally, I define the totality of all logical terms over the Group as the totality of predicates, relations, and quantifiers corresponding to all distinct 10-operators. Generalizing, I define the totality of logical terms as functions that to each cardinal α assign an α-operator.

A formal account

First, let me make some preliminary remarks. In the foregoing definitions I use the variable α to range over cardinals identified with equipollent least ordinals. But while I take a cardinal α to be a set of ordinals, I require that the ordinals in α are themselves not sets of ordinals. This requirement is introduced to ensure that "the index image of x," defined below, is one-to-one. (We can treat ordinals as individuals, we can replace each von Neumann ordinal α with the pair (β, a), where a is some fixed object, etc.) Throughout the book I use lowercase Greek letters $\alpha, \beta, \gamma, \delta, \ldots$ both as variables ranging over cardinals and as variables ranging over ordinals. It is always clear from the context what the range of a given variable is.

I identify a 1-tuple with its member, i.e., $\langle x\rangle = x$.

In earlier chapters I often distinguished between predicates (1 place) and relations (many places). Below I will talk only about predicates, referring to relations as many-place predicates.

DEFINITION 1 Let A be a set indexed by $\alpha = |A|$, where an indexing of A by α is a one-to-one function from α onto A. The *index image* of x, $i(x)$, under the given indexing is as follows:

- If $x \in A$, $i(x) = (\iota\beta \in \alpha)(x = a_\beta)$.
- If $x \subseteq A^n$, $n \geq 1$, $i(x) = \{\langle \beta_1, \ldots, \beta_n\rangle \in \alpha^n\colon \langle a_{\beta_1}, \ldots, a_{\beta_n}\rangle \in x\}$.

TERMINOLOGY Let α be a cardinal number. An *α-individual* is a member of α; an n-place *α-predicate* is a subset of α^n.

If A is indexed by $\alpha = |A|$, then since the indexing function is one-to-one and onto, an α-individual is the index image of some $a \in A$, and an n-place α-predicate is the index image of some $R \subseteq A^n$, under the given indexing.

DEFINITION 2 Let k be a positive integer. I call $R(\alpha) = \langle r_1(\alpha), \ldots, r_k(\alpha)\rangle$ an *α-predicate-argument* if each $r_i(\alpha)$, $1 \le i \le k$, is an α-individual. I call $R(\alpha) = \langle r_1(\alpha), \ldots, r_k(\alpha)\rangle$ an *α-quantifier-argument* if each $r_i(\alpha)$, $1 \le i \le k$, is either an α-individual or an α-predicate and at least one $r_i(\alpha)$, $1 \le i \le k$, is an α-predicate. If $R(\alpha)$ is either an α-predicate-argument or an α-quantifier-argument, I say that $R(\alpha)$ is an *α-argument*.

Below I categorize various kinds of entities into "types." To simplify the type notation, I use two systems of categorization. Entities categorized by the first system will be said to have *marks*, and entities categorized by the second system will be said to have *types*. An entity with a type is a function, and its *type* is essentially the *mark* (sequence of *marks*) of its argument(s).

DEFINITION 3 A *type* is a sequence of natural numbers, $\langle t_1, \ldots, t_k\rangle$, $k > 0$. A *mark* is also a sequence of natural numbers, $\langle m_1, \ldots, m_k\rangle$, $k > 0$.

CONVENTION If p is the k-tuple

$$\underbrace{\langle 0, \ldots, 0\rangle}_{k \text{ times}},$$

I say that $p = 0^k$. If $p = 0^1$, I say that $p = 0$.

DEFINITION 4 Let $R(\alpha) = \langle r_1(\alpha), \ldots, r_k(\alpha)\rangle$ be an α-argument. The *mark* of $R(\alpha)$, $m_R(\alpha)$, is a k-tuple, $\langle m_1, \ldots, m_k\rangle$, where for $1 \le i \le k$,

$$m_i = \begin{cases} 0 & \text{if } r_i(\alpha) \text{ is an } \alpha\text{-individual,} \\ n & \text{if } r_i(\alpha) \text{ is an } n\text{-place } \alpha\text{-predicate.} \end{cases}$$

DEFINITION 5 Let $R_1(\alpha)$, $R_2(\alpha)$ be two α-arguments. $R_1(\alpha)$ and $R_2(\alpha)$ are *similar* iff for some permutation m of α, $R_1(\alpha) = m(R_2(\alpha))$, where $m(R_2(\alpha))$ is the image of $R_2(\alpha)$ under the map induced by m (which I also symbolize by m).

TERMINOLOGY If $R(\alpha)$ is an α-argument, I designate the equivalence class of $R(\alpha)$ under the relation of similarity, defined above, as $[R(\alpha)]$. I call $[R(\alpha)]$ a *generalized α-argument*. If $R(\alpha)$ is of mark p, I say that $[R(\alpha)]$ is also of mark p. I call a set of generalized α-arguments an *α-structure*.

DEFINITION 6 Let $[\mathfrak{R}(\alpha)]$ be the set of all generalized α-arguments of a given mark. An *α-operator* is a function

$$o_\alpha : [\mathfrak{R}(\alpha)] \to \{\mathrm{T}, \mathrm{F}\}.$$

If $[\mathfrak{R}(\alpha)]$ is a set of generalized α-predicate-arguments, I call o_α an *α-predicate-operator*; if $[\mathfrak{R}(\alpha)]$ is a set of generalized α-quantifier-arguments, I call o_α an *α-quantifier*. If the members of $[\mathfrak{R}(\alpha)]$ are of mark p, I say that o_α is of *type p*. We can identify an α-operator with an α-structure, namely the set of all $[R(\alpha)]$'s in its domain such that $o([R(\alpha)]) = \mathrm{T}$.

To prove one-to-one correspondence between α-operators and logical predicates and quantifiers of UL restricted to $\mathfrak{A}(|\mathfrak{A}| = \alpha)$, we need a few additional definitions.

DEFINITION 7 If C is a logical predicate or quantifier satisfying conditions (A) to (E) of chapter 3, then *the restriction of C to $\mathfrak{A}$*, $C_{\mathfrak{A}}$, is as follows: Let $f_c(\mathfrak{A})$ be as in chapter 3, section 6. If $f_c(\mathfrak{A})$ is a subset of $B_1 \times \cdots \times B_k$ (see condition (C)), then $C_{\mathfrak{A}}$ is a function from $B_1 \times \cdots \times B_k$ into $\{\mathrm{T}, \mathrm{F}\}$ such that $C_{\mathfrak{A}}(x_1, \ldots, x_k) = \mathrm{T}$ iff $\langle x_1, \ldots, x_k\rangle \in f_c(\mathfrak{A})$.

DEFINITION 8 Let A be a set. If $x \in A$, then the *mark* of x is 0. If $x \subseteq A^n$, $n > 0$, the *mark* of x is n.

DEFINITION 9 Let $\mathfrak{A}$ be a model with universe A.

- If C is a k-place logical predicate, then the *type* of $C_{\mathfrak{A}}$ is

$$\underbrace{\langle 0, \ldots, 0\rangle}_{k \text{ times}} = 0^k.$$

- If C is a k-place logical quantifier and $x = \langle x_1, \ldots, x_k\rangle \in \mathrm{Dom}(C_{\mathfrak{A}})$, then the *type* of $C_{\mathfrak{A}}$ is $\langle t_1, \ldots, t_k\rangle$, where for $1 \le i \le k$, t_i is the mark of x_i (see definition 8).

I sum up the mark/type classification in table 4.1.

I now state a theorem establishing a one-to-one correspondence between α-operators and logical predicates and quantifiers of UL restricted to an arbitrary model $\mathfrak{A}$ of cardinality α.

THEOREM 1 Let $\mathfrak{A}$ be a model with a universe A of cardinality α. Let $\mathscr{C}|\mathfrak{A}$ be the set of all logical predicates and quantifiers of UL restricted to $\mathfrak{A}$. Let $\mathscr{O}_\alpha$ be the set of all α-operators. Then there exists a 1-1 function h from

Table 4.1
The mark/type classification

Mark	Type
α-individual: 0	k-place α-predicate operator, p_α: 0^k
n-place α-predicate: n	k-place α-quantifier, $q_\alpha : \langle t_1, \ldots, t^k \rangle$*
$x \in A : 0$	k-place logical predicate, $P_{\mathfrak{A}} : 0^k$
$x \subseteq A^n : n$	k-place logical quantifier, $Q_{\mathfrak{A}} : \langle t_1, \ldots, t_k \rangle$†

*Here t_i, $1 \leq i \leq k$, is the mark of $r_i(\alpha)$, where $[R(\alpha) = \langle r_1(\alpha), \ldots, r_k(\alpha) \rangle] \in \mathrm{Dom}(q_\alpha)$.
†Here t_i, $1 \leq i \leq k$, is the mark of x_i, where $\langle x_1, \ldots, x_k \rangle \in \mathrm{Dom}(Q_{\mathfrak{A}})$. (I assume that an empty n-place relation has a different mark from an empty m-place relation, where $n \neq m$.)

$\mathcal{O}_\alpha$ onto $\mathscr{C}|\mathfrak{A}$ defined as follows: For every $o_\alpha \in \mathcal{O}_\alpha$, $h(o_\alpha)$ is the logical term $C_{\mathfrak{A}}$ such that

- o_α and $C_{\mathfrak{A}}$ are of the same type;
- if $\langle s_1, \ldots, s_k \rangle$ is a k-tuple in $\mathrm{Dom}(C_{\mathfrak{A}})$, then $C_{\mathfrak{A}}(s_1, \ldots, s_k) = o_\alpha[\langle i(s_1), \ldots, i(s_k) \rangle]$, where for some indexing I of A by α, $i(s_1), \ldots, i(s_k)$ are the index images of $s_1, \ldots, s_k$, respectively, under I.

Proof See the appendix.

I symbolize the α-operator correlated with $C_{\mathfrak{A}}$ as o_α^C.

Let me give a few examples of the α-operators corresponding to logical predicates and quantifiers restricted to an arbitrary model $\mathfrak{A}$ with a universe A of cardinality α. I will define the α-counterparts of the logical predicates and quantifiers of the examples in chapter 3.

(37) The identity relation $I_{\mathfrak{A}}$ corresponds to o_α^I, an α-predicate of type $\langle 0, 0 \rangle$, defined by $o_\alpha^I(X) = \mathrm{T}$ iff for some $\beta \in \alpha$, $X = [\langle \beta, \beta \rangle]$.

(38) The universal quantifier $\forall_{\mathfrak{A}}$ corresponds to $o_\alpha^\forall$, an α-quantifier of type $\langle 1 \rangle$, defined by $o_\alpha^\forall(X) = \mathrm{T}$ iff $X = [\alpha]$.

(39) The existential quantifier $\exists_{\mathfrak{A}}$ corresponds to $o_\alpha^\exists$, an α-quantifier of type $\langle 1 \rangle$, defined by $o_\alpha^\exists(X) = \mathrm{T}$ iff for some $s \subseteq \alpha$ such that $s \neq \varnothing$, $X = [s]$.

(40) The cardinal quantifiers $\mathrm{C}_{\mathfrak{A}}$ correspond to o_α^δ, α-quantifiers of type $\langle 1 \rangle$, defined by $o_\alpha^\delta(X) = \mathrm{T}$ iff for some $s \subseteq \alpha$ such that $|s| = \delta$, $X = [s]$.

(41) The quantifiers "finitely many" and "uncountably many," $\mathrm{FIN}_{\mathfrak{A}}$ and $\mathrm{UNC}_{\mathfrak{A}}$, correspond to o_α^{FIN} and o_α^{UNC}, α-quantifiers of type $\langle 1 \rangle$, defined by $o_\alpha^{\mathrm{FIN}}(X) = \mathrm{T}$ iff for some $s \subseteq \alpha$ such that $|s| < \aleph_0$, $X = [s]$; $o_\alpha^{\mathrm{UNC}}(X) = \mathrm{T}$ iff for some $s \subseteq \alpha$ such that $|x| > \aleph_0$, $X = [s]$.

(42) The quantifier "as many as not," $\mathrm{MN}_{\mathfrak{A}}$, corresponds to o_α^{MN}, an α-quantifier of type $\langle 1 \rangle$, defined by $o_\alpha^{\mathrm{MN}}(X) = T$ iff for some $s \subseteq \alpha$ such that $|s| \geq |\alpha - s|$, $X = [s]$.

(43) The 1-place quantifier "most," $\mathrm{M}^1_{\mathfrak{A}}$, corresponds to $o_\alpha^{\mathrm{M}^1}$, an α-quantifier of type $\langle 1 \rangle$, defined by $o_\alpha^{\mathrm{M}^1}(X) = \mathrm{T}$ iff for some $s \subseteq \alpha$ such that $|s| > |\alpha - s|$, $X = [s]$.

(44) The 2-place quantifier "most," $\mathrm{M}^{1,1}_{\mathfrak{A}}$, corresponds to $o_\alpha^{\mathrm{M}^{1,1}}$, an α-quantifier of type $\langle 1, 1 \rangle$, defined by $o_\alpha^{\mathrm{M}^{1,1}}(X) = \mathrm{T}$ iff for some $s, t \subseteq \alpha$ such that $|s \cap t| > |s - t|$, $X = [\langle s, t \rangle]$.

(45) The 1-place "well-ordering" quantifier $\mathrm{WO}_{\mathfrak{A}}$ corresponds to o_α^{WO}, an α-quantifier of type $\langle 2 \rangle$, defined by $o_\alpha^{\mathrm{WO}}(X) = \mathrm{T}$ iff for some $r \subseteq \alpha^2$ such that r well-orders $\mathrm{Fld}(R)$, $X = [r]$.

(46) The (second-level) set-membership quantifier $\mathrm{SM}_{\mathfrak{A}}$ corresponds to o_α^{SM}, an α-quantifier of type $\langle 0, 1 \rangle$, defined by $o_\alpha^{\mathrm{SM}}(X) = \mathrm{T}$ iff for some $\beta \in \alpha$ and $s \subseteq \alpha$ such that $\beta \in s$, $X = [\langle \beta, s \rangle]$.

(47) The quantifier "ordering of the natural numbers with zero," $\mathrm{NZ}_{\mathfrak{A}}$, corresponds to o_α^{NZ}, an α-quantifier of type $\langle 2, 0 \rangle$, defined by $o_\alpha^{\mathrm{NZ}}(X) = \mathrm{T}$ iff for some $r \subseteq \alpha^2$ and $\beta \in \alpha$ such that $\langle \mathrm{Fld}(r), r, \beta \rangle \cong \langle \omega, <, 0 \rangle$, $X = [\langle r, \beta \rangle]$.

(48) The "the" quantifier, $\mathrm{THE}_{\mathfrak{A}}$, corresponds to o_α^{THE}, an α-quantifier of type $\langle 1, 1 \rangle$, defined by $o_\alpha^{\mathrm{THE}}(X) = \mathrm{T}$ iff for some $s, t \subseteq \alpha$ such that $|s| = 1$ and $s \subseteq t$, $X = [\langle s, t \rangle]$.

I define logical operators as follows:

DEFINITION 10 A *logical operator* of type t is a function that assigns to each cardinal α an α-operator of type t.

3 Unrestricted First-Order Logic: Syntax and Semantics

I can now delineate the syntax and the semantics of first-order logic with Tarskian logical terms satisfying the metatheoretical requirements specified in chapter 3 and defined by means of *logical operators*. As before, I will leave logical functors and quantifier functors out for the sake of simplicity.

Syntax

Let me first present the preliminary notion of the *type of a constant*. A type t is, recall, a sequence of natural numbers $\langle t_1, \ldots, t_k \rangle$, where k is a positive integer. Intuitively, the type of a constant gives us information about its arguments.

- Individual constants do not have a type (since they do not have arguments).
- The type of logical and nonlogical k-place first-level predicates is $\underbrace{\langle 0, \ldots, 0 \rangle}_{k \text{ times}} = 0^k$.
- The type of k-place quantifiers is $\langle t_1, \ldots, t_k \rangle$, where for some $1 \le i \le n$, $k_i > 0$. (Intuitively, if the ith argument of a k-place quantifier Q is a singular term, $t_i = 0$; if the ith argument is an n-place first-level predicate, $t_i = n$.)

Primitive symbols

1. Logical symbols
   a. sentential connectives: any collection that semantically forms a complete system of truth-functional connectives, say $\sim$, &, $\vee$, $\rightarrow$, $\leftrightarrow$
   b. n logical predicates and/or quantifiers, $C_1, \ldots, C_n$ of types $t_1, \ldots, t_n$ respectively, $n > 0$
2. Variables: $x_1, x_2, \ldots$ (informally: x, y, z, v, w)
3. Punctuation symbols: (a) parentheses: (,); (b) comma: ,.
4. Nonlogical symbols
   a. individual constants: $a_1, \ldots, a_m$, $m \ge 0$
   b. predicate constants: for each $n > 0$, a finite (possibly empty) set of n-place predicates, $P_1^n, \ldots, P_m^n$.

Well-formed formulas (wffs)

1. Terms: Individual constants and variables are terms.
2. Atomic wffs: If S is an n-place predicate (logical or nonlogical) and $s_1, \ldots, s_n$ are terms, then $S(s_1, \ldots, s_n)$ is an atomic wff.
3. Wffs
   a. An atomic wff is a wff.
   b. If Φ, Ψ are wffs, then so are $(\sim\Phi)$, $(\Phi \,\&\, \Psi)$, $(\Phi \vee \Psi)$, $(\Phi \rightarrow \Psi)$ and $(\Phi \leftrightarrow \Psi)$.
   c. If Q is a quantifier of type $t = \langle t_1, \ldots, t_k \rangle$, n is the maximum of $\{t_1, \ldots, t_k\}$, $x_1, \ldots, x_n$ are distinct variables, and $B_1, \ldots, B_k$ are expressions such that for each $1 \le i \le k$, if $t_i = 0$, B_i is a term and otherwise B_i is a wff, then $((\mathrm{Q}x_1, \ldots, x_n)(B_1, \ldots, B_k))$ is a wff.

I follow the convention that outermost parentheses in wffs may be omitted.

Bound and free occurrences of variables in wffs I say that x occurs in an expression e iff either $x = e$ or x is a member of the sequence of primitive symbols constituting e.

- There are no bound occurrences of variables in terms.
- If Φ is an atomic wff, then no occurrence of x in Φ is bound.
- If Φ is a wff of the form $\sim\Psi$, then an occurrence of x in Φ is bound iff it is bound in Ψ.
- If Φ is a wff of the form $\Psi \,\&\, \Xi$, $\Psi \vee \Xi$, $\Psi \rightarrow \Xi$, or $\Psi \leftrightarrow \Xi$, then an occurrence of x in Φ is bound iff it is either a bound occurrence in Ψ or it is a bound occurrence in Ξ.
- If Φ is a wff of the form $(\mathrm{Q}x_1, \ldots, x_n)(B_1, \ldots, B_k)$, where Q is of type $\langle t_1, \ldots, t_k \rangle$, then an occurrence of x in Φ is bound iff it is an occurrence in some B_i, $1 \le i \le k$, such that either x is bound in B_i or for some $1 \le m \le t_i$, $x = x_m$.
- An occurrence of x in Φ is free iff it is not bound.

The idea is that if Q is, say, of type $\langle 1, 2, 0 \rangle$ and R_1, R_2 are two 2-place predicates of the language, then in the wff

$(\mathrm{Q}x, y)(R_1(x, y), R_2(x, y), x)$

Q binds the first two occurrences of x and the second occurrence of y, but the third occurrence of x and the first occurrence of y are free. To make the notation more transparent, I sometimes indicate the type of a quantifier Q with a superscript. That involves rewriting the formula above, for example, as

$(\mathrm{Q}^{1,2,0}x, y)(R_1(x, y), R_2(x, y), x)$.

Sentences A *sentence* is a wff in which no variable occurs free.

In practice I will sometimes omit commas separating the variables in a quantifier expression. Thus instead of $(\mathrm{Q}x, y)$, I will write $(\mathrm{Q}xy)$. I will also use various types of parentheses.

Semantics

Let $\mathscr{L}$ be a first-order logic with syntax as defined above. Say $\mathscr{L}$ has logical predicates [illegible] logical quantifiers $Q_1, \ldots, Q_{[illegible]}$, nonlogical [illegible]

P_{n_4}. Each logical predicate or quantifier C of type t is semantically defined by means of a logical operator o^C of the same type.

Let $\mathfrak{A}$ be a model for the language with universe A of cardinality α, defined relative to the nonlogical vocabulary of $\mathscr{L}$ in the usual way. That is, $\mathfrak{A} = \langle A, a_1^{\mathfrak{A}}, \ldots, a_{n_3}^{\mathfrak{A}}, P_1^{\mathfrak{A}}, \ldots, P_{n_4}^{\mathfrak{A}} \rangle$. Let g be an assignment of elements in A to the variables of the language. I define an extension of g, $\bar{g}$, to the terms of the language as follows: For a variable x, $\bar{g}(x) = g(x)$. For an individual constant a, $\bar{g}(a) = a^{\mathfrak{A}}$.

DEFINITION OF SATISFACTION $\mathfrak{A}$ satisfies the wff Φ with the assignment g—$\mathfrak{A} \models \Phi[g]$—iff the following conditions hold:

1. Atomic wffs
   a. Let P be an n-place nonlogical predicate and $s_1, \ldots, s_n$ terms. Then

      $\mathfrak{A} \models P(s_1, \ldots, s_n)[g]$ iff $\langle \bar{g}(s_1), \ldots, \bar{g}(s_n) \rangle \in P^{\mathfrak{A}}$.

      (As before, I identify a 1-tuple with its member.)
   b. Let V be an n-place logical predicate and $s_1, \ldots, s_n$ terms. Then

      $\mathfrak{A} \models V(s_1, \ldots, s_n)[g]$ iff there is an indexing I of A by α such that $o_\alpha^V[\langle i(\bar{g}(s_1)), \ldots, i(\bar{g}(s_n)) \rangle] = \mathrm{T}$,

      where for $1 \leq j \leq n$, $i(\bar{g}(s_j))$ is the index image of $\bar{g}(s_j)$ under I. (See definition 1.)
2. Nonatomic wffs
   a. Let Φ, Ψ be wffs.

      $\mathfrak{A} \models {\sim}\Phi[g]$ iff $\mathfrak{A} \not\models \Phi[g]$;
      $\mathfrak{A} \models (\Phi \,\&\, \Psi)[g]$ iff $\mathfrak{A} \models \Phi[g]$ and $\mathfrak{A} \models \Psi[g]$

      ...
   b. Let Q be a quantifier of type $\langle t_1, \ldots, t_k \rangle$, let n be the maximum of $\{t_1, \ldots, t_k\}$, let $x_1, \ldots, x_n$ be distinct variables, and let $B_1, \ldots, B_k$ be expressions such that for each $1 \leq j \leq k$, if $t_j = 0$, B_j is a term, and otherwise B_j is a wff. Then

      $\mathfrak{A} \models (\mathrm{Q}x_1, \ldots, x_n)(B_1, \ldots, B_k)[g]$ iff there is an indexing I of A by α such that $o_\alpha^{\mathrm{Q}}[\langle i(\bar{g}_{\bar{x}_{t_1}}(B_1)), \ldots, i(\bar{g}_{\bar{x}_{t_k}}(B_k)) \rangle] = \mathrm{T}$,

      where for $1 \leq j \leq k$,

      if $t_j = 0$, then $\bar{g}_{\bar{x}_{t_j}}(B_j) = \bar{g}(B_j)$;
      if $t_j \geq 1$, $g_{\bar{x}_{t_j}}(B_j) = \{\langle a_1, \ldots, a_{t_j} \rangle \in A^{t_j} : \mathfrak{A} \models$
      $B_j[g(x_1 \ldots$ [illegible]

DEFINITION OF TRUTH IN A MODEL Let $\mathscr{L}$ and $\mathfrak{A}$ be as above. Let Φ be a sentence of $\mathscr{L}$. Then Φ is *true* in $\mathfrak{A}$—$\mathfrak{A} \models \Phi$—iff for some assignment g of elements in A to the variables of the language, $\mathfrak{A} \models \Phi[g]$.

Examples Let $\mathfrak{A}$ be a model with the Group as its universe. Let P and M be the 1-place predicates "x is a philosopher" and "x is a mathematician" respectively, $P^{\mathfrak{A}} = \{\text{Alan}\}$, and $M^{\mathfrak{A}} = \{\text{Alan, Jane}\}$. Let G be the 2-place relation "x graduated from Columbia College before y" and $G^{\mathfrak{A}} = \{\langle \text{Ian, Carl} \rangle, \langle \text{Carl, Gary} \rangle, \langle \text{Ian, Gary} \rangle\}$. The quantifiers !1 ("there is exactly one"), $\mathrm{M}^{1,1}$("most ____ are..."), and TL-F ("three individuals stand in the linear relation ____, the first being..."), restricted to $\mathfrak{A}$, are definable by the following 10-operators:

(49) $o_{10}^{!1} : \mathrm{EQ}(\boldsymbol{P}(10)) \rightarrow \{\mathrm{T, F}\}$, where $o_{10}^{!1}[X] = \mathrm{T}$ iff X is similar to $\{0\}$.

(50) $o_{10}^{\mathrm{M}^{1,1}} : \mathrm{EQ}(\boldsymbol{P}(10) \times \boldsymbol{P}(10)) \rightarrow \{\mathrm{T, F}\}$, where $o_{10}^{\mathrm{M}^{1,1}}[\langle X, Y \rangle] = \mathrm{T}$ iff $|X \cap Y| > |X - Y|$.

(51) $o_{10}^{\mathrm{TL-F}} \mathrm{EQ}(\boldsymbol{P}(10^2) \times 10) \rightarrow \{\mathrm{T, F}\}$, where $o_{10}^{\mathrm{TL-F}}[\langle R, x \rangle] = \mathrm{T}$ iff $\langle R, x \rangle$ is similar to $\langle \{\langle 0, 1 \rangle, \langle 1, 2 \rangle, \langle 0, 2 \rangle\}, 0 \rangle$.

$\mathrm{EQ}(Z)$ is the set of all equivalence classes of members of Z under the relation of similarity.

Let I be any indexing of the Group by 10, say indexing by alphabetical order of members' first names. Then

(52) There is exactly one philosopher,

or formally,

(53) $(!1x)Px$,

is true in $\mathfrak{A}$, since $i(P^{\mathfrak{A}}) = \{0\}$.

(54) There is exactly one mathematician,

or,

(55) $(!1x)Mx$,

is false in $\mathfrak{A}$, since $i(M^{\mathfrak{A}}) = \{0, 9\}$ is not similar to $\{0\}$.

(56) Most philosophers are also mathematicians.

or,

(59) (TL-Fxy)(Gxy, Ian),

is true in $\mathfrak{A}$, because $\langle\{\langle 8, 2\rangle, \langle 2, 6\rangle, \langle 8, 6\rangle\}, 8\rangle$ is similar to $\langle\{\langle 0, 1\rangle, \langle 1, 2\rangle, \langle 0, 2\rangle\}, 0\rangle$.

4 Higginbotham and May's Relational Quantifiers

My characterization of logical terms as logical operators puts all logical predicates and quantifiers on a par. It captures a basic principle of logicality, namely that to be logical is to take only structure into consideration. Also captured is the complementary principle that every structure is mirrored by some logical term. It is, however, interesting to divide the expanse of logical terms into groups according to significant characteristics. Mostowski's work allows us to single out predicative quantiers by identifying a method of individuation particular to these quantifiers. In "Questions, Quantifiers, and Crossing" (1981) J. Higginbotham and R. May distinguish four groups of relational quantifiers of the simplest kind, type $\langle 2\rangle$, by means of the *invariance conditions* they satisfy. Their criterion orders simple relational quantifiers according to their complexity, from quantifiers that can only distinguish the number of pairs a binary relation R contains to "fine-grained" quantifiers that take into account the inner structure of R.

Given a universe A, Higginbotham and May define a binary relational quantifier over A as a function $q : P(A \times A) \rightarrow \{T, F\}$. They consider the following invariance conditions:[1]

a. Invariance under automorphisms of A × A
b. (1) Invariance under 1-automorphisms of A × A
   (2) Invariance under 2-automorphisms of A × A
c. Invariance under pair-automorphisms of A × A
d. Invariance under automorphisms of A

Given a set A, $m : A \times A \rightarrow A \times A$ is a (set) *automorphism* of $A \times A$ iff m is a permutation of $A \times A$.

An automorphism $m : A \times A \rightarrow A \times A$ is a 1-*automorphism* of $A \times A$ iff for all $a, b, a', b', c, d, c', d' \in A$,

$m(a, b) = (a', b')$ and $m(c, d) = (c', d')$ implies ($a = c$ iff $a' = c'$).

That is, m is a 1-automorphism of $A \times A$ iff there is an automorphism m_1 of A such that for all $a, b \in A$,

$m(a, b) = (m_1(a), b')$

for some $b' \in A$. Informally, if p_1 and p_2 are pairs with the same first element, then a 1-automorphism m will assign to p_1 and p_2 pairs that also share their first element. In such a case I will say that m respects first elements.

An automorphism $m : A \times A \rightarrow A \times A$ is a 2-*automorphism* of $A \times A$ iff for all $a, b, a', b', c, d, c', d' \in A$,

$m(a, b) = (a', b')$ and $m(c, d) = (c', d')$ implies ($b = d$ iff $b' = d'$).

That is, m is a 2-automorphism of $A \times A$ iff there is an automorphism m_2 of A such that for all $a, b \in A$,

$m(a, b) = (a', m_2(b))$

for some $a' \in A$. Informally, m respects second elements.

An automorphism $m : A \times A \rightarrow A \times A$ is a *pair-automorphism* of $A \times A$ iff m is both a 1-automorphism of $A \times A$ and a 2-automorphism of $A \times A$. That is, m is a pair-automorphism of $A \times A$ iff there are automorphisms m_1, m_2 of A such that for all $a, b \in A$,

$m(a, b) = (m_1(a), m_2(b))$.

In such a case I will say that m respects both first and second elements.

The invariance conditions (a) to (d) increasingly extend the notion of relational quantifier, with (a) reflecting a minimalist approach and (d) a maximalist approach. All quantifiers satisfying (a), (b), or (c) satisfy (d), but some quantifiers satisfying (d) do not satisfy (a) to (c); some quantifiers satisfying (c) do not satisfy either (b.1) or (b.2), etc. The more invariance conditions a quantifier satisfies, the less distinctive it is. A quantifier satisfying (a), for instance, does not distinguish between relations that have the same number of elements but otherwise differ in structure (for example, the one is a well-ordering relation, while the other is not). Quantifiers satisfying (d) are those for which I developed my "constructive" definition. Ipso facto, all quantifiers satisfying Higginbotham and May's conditions fall under my definition. Let us describe the quantifiers in each of Higginbotham and May's categories.

Invariance condition (a) The relational quantifiers satisfying (a) constitute an immediate extension of Mostowski's quantifiers and are definable by his cardinality functions. These quantifiers treat relations as sets, and elements of relations, i.e., n-tuples of individuals, as individuals. I will call these *weak relational quantifiers*.

The contribution of weak relational quantifiers to the expressive power of first-order logic is straightforward. They allow us to enumerate the elements of relations: "____ pair(s) of individuals in the universe stand(s) in the binary relation R," and similarly for n-place relations. Thus we can define the 1-place weak relational quantifier

(60) $(\text{Most}^1\ xy)Rxy$

("Most pairs of individuals in the universe fall under the relation R") by the same function t that defines the 1-place predicative "most." Similarly, the 2-place relational "most,"

(61) $(\text{Most}^{1,1}\ xy)(Rxy, Sxy)$

("Most pairs standing in the relation R stand in the relation S"), is defined by the same cardinality function as the 2-place predicative "most."

Weak relational quantifiers do not exhaust the cardinality properties of relations, however. Among the cardinality properties not expressible by weak relational quantifiers is the following:

(62) The (binary) relation R has α elements in its domain,

where α is a cardinal number. Instances of (62) can be stated using a pair of predicative quantifiers:

(63) $(!\alpha\ x)(\exists y)Rxy$

But no weak relational quantifier is equivalent to the pair $(!\alpha\ x)(\exists y)$.

Invariance condition (b) The relational quantifiers satisfying invariance condition (b) essentially say how many individuals in the universe stand to how many individuals in a given relation R. The difference between the two conditions (b.1) and (b.2) is in the direction from which the relation is perceived. Quantifiers satisfying the first condition basically say that α objects in the universe are such that each stands in the relation R to β objects in the universe. Quantifiers satisfying the second condition say that there are β objects in the universe to each of which α objects in the universe stand in the relation R. (The properties predicated on relations by quantifiers satisfying (b.1) and (b.2) can be more complex than those described above, but for my purposes it suffices to consider the basic properties.) Since the two conditions under (b) are symmetrical, it is enough to discuss just one. Following Higginbotham and May, I will concentrate on the first. Higginbotham and May prove that all quantifiers satisfying (b) assign cardinality properties to relations in their scope. A detailed description and proof of their claim appears in the appendix.

Intuitively, we arrive at the cardinality counterparts of quantifiers satisfying invariance condition (b.1) in the following way: Given a model $\mathfrak{A}$ with a universe A of cardinality α and a binary relation $R \subseteq A^2$, we can describe R from the point of view of its cardinalities by stating, with respect to each element of A, to how many objects in A it stands in the relation R and to how many objects in A it does not stand in the relation R. We can thus represent the cardinalities of R by means of a function

$$f: \alpha \to (\beta, \gamma)_\alpha,$$

where α serves as a set of indices for the elements of A (as in section 2 above) and $(\beta, \gamma)_\alpha$ is the set of all pairs of cardinals β, γ whose sum is α. Given an element $a_\delta \in A$, $f(\delta)$ is the pair of cardinals $\langle \beta, \gamma \rangle$ such that a_δ stands in the relation R to β individuals and a_δ does not stand in the relation R to γ individuals. But quantifiers do not distinguish which elements of A are associated with a given pair of cardinals $\langle \beta, \gamma \rangle$. Therefore, Higginbotham and May construct equivalence classes of functions f under a similarity relation. Quantifiers are then defined as functions from such equivalence classes to truth values. As you can see, there is a certain resemblance between Higginbotham and May's cardinality functions and my α-operators. Indeed, I arrived at the idea of my definition by generalizing Higginbotham and May's method.

Invariance condition (c) Quantifiers invariant under pair-automorphisms of $A \times A$ distinguish identities and nonidentities both in the domain and in the range of a given relation R. These quantifiers can express such properties of relations having to do with identities as, e.g., "____ is a one-to-one relation."

Invariance condition (d) I will call relational quantifiers satisfying invariance under automorphisms of A, but not the other invariance conditions, *strong relational quantifiers*. Strong relational quantifiers are genuine logical terms, and they can be represented by logical operators defined in section 2 above. These quantifiers make the finest distinctions among relations that logical terms are capable of making. Below I will give several examples of strong relational quantifiers in natural language, and also of weaker relational quantifiers satisfying (a) through (c).

5 Linguistic Applications

Several "types" of logical terms of UL have received ample attention in logico-linguistic circles, usually under the heading of "generalized quantifiers." In chapter 2 we saw Mostowskian quantifiers being used to interpret determiners. In the present section I will further expand the domain of applications of UL quantifiers. My discussion will not assume the form of a survey. Instead, I will describe applications of logical quantifiers that came up in the course of my own investigations. (Other works devoted to linguistic applications of, or theoretical linguistic approaches to, generalized quantifiers are listed in the references. The reader is referred to Barwise and Cooper, Higginbotham and May, Keenan, Keenan and Moss, Keenan and Stavi, May, van Benthem, and Westerståhl, among others.)

I will begin with a new application of Mostowskian quantifiers and then proceed through Higginbotham and May's categories to describe increasingly strong relational quantifiers in natural language.

Generalized operations on relations

In standard first-order logic we use the existential and universal quantifiers as operators that, given two binary relations R and S, yield new relations called the relative product of R and S— $R|^P S$ —and the relative sum of R and S— $R|^S S$. These are defined (by dual conditions) as follows:

$$R|^P S =_{df} \{\langle x, y\rangle : (\exists z)(xRz \,\&\, zSy)\}$$

$$R|^S S =_{df} \{\langle x, y\rangle : (\forall z)(xRz \vee zSy)\}$$

Linguistically, we can interpret the relation "being a paternal uncle of" as the relative product of the relations "being a brother of" and "being a father of," etc. By generalizing the definitions of relative product and sum, we arrive at the notion of a relative product/sum modulo Q, where Q is a 1-place Mostowskian quantifier. I define the relative product and sum of binary relations R and S modulo Q as follows:

$$R|^P_Q S =_{df} \{\langle x, y\rangle : (Qz)(xRz \,\&\, zSy)\}$$

$$R|^S_Q S =_{df} \{\langle x, y\rangle : (Qz)(xRz \vee zSy)\}$$

(As in the traditional product and sum, if Q_1 is the dual of Q_2, the definiens of $R|^P_{Q_1} S$ is the dual of the definiens of $R|^S_{Q_2} S$.) I will call the standard relative product the relative product modulo $\exists$ and the standard relative sum the relative sum modulo $\forall$. The notions of relative product and sum allow us to define relations that include a "cardinality factor." The operation of relative product modulo Q appears to be especially useful, as can be seen in the following examples:

(64) x is a friend of many people who know y.

(65) x has few common acquaintances with y.

When R is an ordering relation, we can define relations that have to do with distance or relative position in R as relative products of R modulo

the appropriate Q. In this way we can define

(66) There are n elements between x and y in R.

(67) x is far behind/ahead-of y in R.

(68) x is second best to y in P.

Here P is a property (e.g., diving) that determines the field of an implicit ordering relation R, "being better at . . . ."

Two-place predicative quantifiers can also be used to define sets and relations that include a cardinality factor. I call the operation of constructing such a set (or relation) from two initial relations R and R' "a generalized relative product of R and R'." For example, using the quantifier "same number," defined in the obvious way, we can single out the median element in a linear ordering relation with

(69) (same-number z)(xRz, zRx).

In a similar way, we can define "x is relatively high/low in R."

It is often useful to consider "semilinear" orderings, an ordering like a linear ordering but with the requirement "$(\forall x)(\forall y)(x < y \vee y > x \vee x = y)$" replaced by "$(\forall x)(\forall y)(x < y \vee y > x \vee x \approx y)$," where $\approx$ is some equivalence relation, for example "being in the same income bracket as." Thus if R is a semilinear ordering relative to "being in the same income bracket as," (69) will give us the set of all elements in the middle income bracket. Using a second predicative quantifier, we can now express statements indicating how many individuals occupy a certain relative position in R. For example,

(70) Proportionally more women hold high-paying jobs in San Diego than in other cities in the country.

Other statements stating formal properties of generalized relative products of R and S can be constructed using relational quantifiers defined in this chapter.

Weak relational quantifiers

I will indicate some of the uses of weak relational quantifiers. Given a relative product modulo Q, e.g., (66), we can use weak relational quantifiers to make statements of the form

(71) There are m pairs whose distance in R is n.

Other cases of quantification where pairs are taken as basic units are also naturally expressed using weak relational quantifiers. For example,

(72) Most divorced couples do not remarry.

Consider, however,

(73) Four married couples left the party.

The most natural construal of (73) as a weak relational quantification fails. Suppose that "exactly 4," !4, is a 2-place weak relational quantifier over binary relations. Then, since !4 is essentially a Mostowskian quantifier, we can define it by a cardinality function as described in chapter 2. That is, given a universe A, $t_A^{!4}$ is a function such that for any quadruple $\alpha, \beta, \gamma, \delta$, where $\alpha + \beta + \gamma + \delta = |A|$,

$t_A^{!4}(\alpha, \beta, \gamma, \delta) = \mathrm{T}$ iff $\alpha = 4$.

This means that if R and S are binary relations on A,

$!4(R, S) = \mathrm{T}$ iff $|R \cap S| = 4$.

Now, if we interpret (73) as

(74) $(!4\ xy)$(x is married to y, x and y left the party),

then (74) turns out true when the number of married couples who left the party is two, not four. (This is because there are two pairs in a couple.) Thus (74) is an incorrect rendering of (73). There are various remedies to the problem. Among them are the following:

a. We can treat binary relations as *sets of couples* (a couple being an unordered pair) and then define weak relational quantifiers as regular Mostowskian quantifiers by setting numerical conditions on the atoms of the Boolean algebra generated by n-tuples of such "sets" in a given universe A. The couple quantifier !4 will thus be defined by the same t-function as the corresponding quantifier based on pairs: $!4(R, S) = \mathrm{T}$ iff the intersection of the two sets of couples R and S yields a set of 4 couples.
b. We can construe couple quantifiers as strong relational quantifiers, i.e., quantifiers satisfying invariance condition (d).

By adopting strategy (a), we will be able to use weak relational quantifiers to symbolize the following English sentences:

(75) Half the students in my class do not know each other.

(76) Most of my friends have few common acquaintances.

(77) Few townsmen and villagers hate each other.

(78) Almost all brothers compete with each other.

Thus, for instance, (75) will be symbolized as

(79) (Half xy)[x is a student in my class & y is a student in my class & $x \neq y$, $\sim$(x knows y & y knows x)].

But to interpret

(80) Most younger brothers envy their elder brothers

we must go back to quantifiers based on pairs.

I should say that weak relational quantifiers (based on pairs or on couples) do not exhaust the possibilities of interpretation of the sentences in our examples. On my interpretation, (75), for instance, is true if my class consists of four students, a, b, c, and d, and one of the students, say a, does not know (and is not known to) anyone in the class, but the rest—b, c, and d—all know each other. Someone may wish to interpret (75) so that it will come out false in the situation just described. This can be done by adopting stronger relational quantifiers.

Linearity quantifiers

Higginbotham and May's 1-place relational cardinality quantifiers over a universe A, i.e., 1-place binary quantifiers invariant under 1-automorphism of $A \times A$, essentially say how many individuals stand to how many individuals in a given binary relation R. But this is exactly what a linear quantifier prefix with two 1-place predicative quantifiers says about a relation R in a model $\mathfrak{A}$ with a universe A. For that reason I name relational cardinality quantifiers *linearity quantifiers*. Higginbotham and May called the operation of constructing a relational quantifier equivalent to a linear quantifier prefix with two predicative quantifiers *absorption*. A relational quantifier constructed by absorption is said to be *separable*.

The rule of absorption is this: if Q_1 and Q_2 are two 1-place predicative quantifiers over a universe A and R is a binary relation included in A^2, then the quantifier prefix $(Q_1 x)(Q_2 y)$ will be absorbed by $(Q_3 xy)$, where Q_3 is a linearity quantifier over A such that

$Q_3(R) = \mathrm{T}$ iff $Q_1(\{a \in A : Q_2(\{b \in A : aRb\}) = \mathrm{T}\}) = \mathrm{T}$.

We can generalize the operation of absorption to n-place quantifier prefixes by defining 1-place linearity quantifiers on n-place relations over a universe A. A 1-place linearity quantifier on an n-place relation over a universe A is a function

$q : P(A^n) \to \{\mathrm{T}, \mathrm{F}\}$

that is invariant under *linear automorphisms* of A^n. I define "linear automorphism of A^n" as follows. The function

$m : A^n \to A^n$

is a linear automorphism of A^n iff m is an automorphism of A^n and for any $a_1, a_2, \ldots, a_n, a_1', a_2', \ldots, a_n', b_1, b_2, \ldots, b_n, b_1', b_2', \ldots, b_n' \in A$ the following holds:

If $m(a_1, a_2, \ldots, a_n) = (a'_1, a'_2, \ldots, a'_n)$ and $m(b_1, b_2, \ldots, b_n) = (b'_1, b'_2, \ldots, b'_n)$, then

1. $a_1 = b_1$ iff $a'_1 = b'_1$, and
2. if $a_1 = b_1$, then $a_2 = b_2$ iff $a'_2 = b'_2$, and

$\vdots$

$n - 1$. if $a_1 = b_1$ and $a_2 = b_2$ and ... and $a_{n-2} = b_{n-2}$, then $a_{n-1} = b_{n-1}$ iff $a'_{n-1} = b'_{n-1}$.

To return to absorption of two linearly ordered 1-place predicative quantifiers, let A be a set of n children, $n \geq 3$. Consider the sentence

(81) Three children had three friends each.

We can formalize (81) with either (82) or (83) below:

(82) $(!3\ x)(!3\ y)$ x is a friend of y.

Here !3 is a 1-place predicative quantifier defined, for A, by a Mostowskian function t such that for any (k, m) in its domain $(k + m = n)$, $t(k, m) = \mathrm{T}$ iff $k = 3$.

(83) $(3/3\ xy)$ x is a friend of y.

Here 3/3 is a linearity quantifier of type $\langle 2 \rangle$ defined, for A, by a Higginbothtam-May function k such that for any $[f]$ in its domain $(f : n \to (j, k)_n)$, $k[f] = \mathrm{T}$ iff f is similar to some f^* such that

$f^*(0) = f^*(1) = f^*(2) = (3, n - 3)$ and

$f^*(3), \ldots, f^*(n - 1) \neq (3, n - 3)$.

Intuitively, the function f^* assigns 3 to children 0, 1, and 2 as the number of their friends, and $n - 3$ as the number of their nonfriends. To all other children f^* assigns a different combination of numbers of friends and nonfriends. (For the sake of simplicity I assumed that a child can have himself or herself as a friend.)

Note, however, that linearity quantifiers on binary relations can also express Boolean combinations, possibly infinite, of linear quantifier prefixes with predicative quantifiers. Thus, consider the following infinite conjunction in which "number" stands for "natural number" and n ranges over the natural numbers:

(84) One number has no predecessors, and two numbers have at most one predecessor, and three numbers have at most two predecessors, and ..., and n numbers have at most $n - 1$ predecessors, and ...

This infinite conjuction cannot be formalized in first-order logic with predicative quantifiers, but it can be formalized in first-order logic with linearity quantifiers on binary relations. I will symbolize it as

(85) (n at most $(n - 1)$ xy)x has a predecessor y,

where "n at most $(n - 1)$" is defined, in a universe A of cardinality $\aleph_0$, by a function

$k : [\mathrm{F}] \to \{\mathrm{T}, \mathrm{F}\}$

such that for any $[f] \in [\mathrm{F}]$, $k[f] = \mathrm{T}$ iff f is similar to the function

$f^* : \aleph_0 \to (k, l)_{\aleph_0}$,

which is such that for every $n < \aleph_0$,

$f^*(n) = (n, \aleph_0)$.

Intuitively, f represents a relation R with field of cardinality $\aleph_0$ such that under some indexing of the universe A by $\aleph_0$, a_0 stands in the relation R to no objects in A, a_1 stands in the relation R to one object in A, a_2 stands in the relation R to two objects in A, and so on. Clearly, k also defines the complex quantifier in (86):

(86) One number has no predecessor, and one number has exactly one predecessor, and one number has exactly two predecessors, and ..., and one number has exactly n predecessors, and ...

Note that k need not express a condition which exhibits a regularity. Using a quantifier k_1 similar (in the intuitive sense) to k, we can represent an irregular situation like the following:

(87) Two children have two friends each, and ten children have four friends each, and twelve children have nine friends each, and ...

Another kind of cardinality condition expressible with linearity quantifiers, but not with a standard prefix of two 1-place predicative quantifiers, is exemplified by the following sentence:

(88) There is a great variance in the number of friends of each of these youngsters

(which could also be phrased as "These youngsters differ considerably in the numbers of their friends"). Assuming, for simplicity, that the universe consists of "these youngsters" and that the friends in question are members of the universe, (88) could be expressed as

(89) (Great variance xy) youngster x has youngster y for a friend,

where for each universe A of cardinality α, "great variance" is defined by the function k such that for every $[f] \in \mathrm{Dom}(k)$,

$k([f]) = \mathrm{T}$ iff there is a wide distribution of cardinals γ

such that for some $\beta \in \alpha, f(\beta) = (\gamma, \alpha - \gamma)$.

We can construct 2-place linearity quantifiers, of type $\langle 1, 2\rangle$, that will enable us to restrict linear quantification to $B \upharpoonright R$ (R with its domain limited to B). If we want to symbolize the following sentence without assuming the universe consists of "these youngsters," we will use the 2-place "great variance" quantifier.

(90) There is a great variance in the number of words in the active vocabulary of each of these youngsters.

This sentence will be rendered "(Great variance xy)(x is one of these youngsters, x has word y in his active vocabulary)."

Let us now turn to absorption of two 2-place predicative quantifiers. A linguistically interesting case is that of quantifications of the form

(91) $(Q_1 x)(\Phi, (Q_2 y)(\Psi, \Xi))$,

where Φ, Ψ, Ξ are well-formed formulas. The quantifiers in (91) are absorbed by the quantifier $(Q_1/Q_2)^{1,2,2}$, defined, for a universe A, as follows: for every $B \subseteq A$ and $C, D \subseteq A^2$,

$(Q_1/Q_2)_A(B, C, D) = \mathrm{T}$ iff $(Q_1)_A(\{a \in A : a \in B\},$
$\{a \in A : (Q_2)_A(\{b \in A : \langle a, b\rangle \in C\}, \{b \in A : \langle a, b\rangle \in D\}) = \mathrm{T}\}) = \mathrm{T}$.

It is easy to see that (91) is equivalent to

(92) $((Q_1/Q_2)^{1,2,2}xy)(\Phi, \Psi, \Xi)$,

whose satisfaction condition in a model $\mathfrak{A}$ with a universe A by an assignment g is

$\mathfrak{A} \models (Q_1/Q_2 xy)(\Phi, \Psi, \Xi)[g]$ iff $(Q_1)_A(\{a \in A : \mathfrak{A} \models \Phi[g(x/a)]\},$
$\{a \in A : (Q_2)_A(\{b \in A : \mathfrak{A} \models \Psi[g(x/a)(y/b)]\},$
$\{b \in A : \mathfrak{A} \models \Xi[g(x/a)(y/b)]\}) = \mathrm{T}\}) = \mathrm{T}$.

This definition of absorption is similar to one proposed by R. Clark and E. L. Keenan in "The Absorption Operator and Universal Grammar" (1986). But there is an essential difference: whereas I constructed the absorption quantifier Q_1/Q_2 in such a way that in the formula

$(Q_1/Q_2)(\Phi x, \Psi xy, \Xi xy)$

Q_1/Q_2 binds all free variables, Clark and Keenan defined Q_1/Q_2 in such

a way that it does not bind the occurrence of x in Ψxy. The reason the absorbing quantifier has to bind x in Ψxy is simple: Q_1/Q_2 has to be so defined that

(93) $(Q_1/Q_2 xy)(\Phi, \Psi, \Xi)$

is logically equivalent to

(94) $(Q_1 x)(\Phi, (Q_2 y)(\Psi, \Xi))$,

no matter what well-formed formulas Φ, Ψ, and Ξ are. Now it is an essential feature of (94) that any free occurrence of x in Φ, Ψ, or Ξ is bound by Q_1, and similarly, that any free occurrence of y in Ψ or Ξ is bound by Q_2. The relation of binding between quantifiers and free variables in (94) must be preserved by (93). In particular, if x occurs free in Ψ, it should be bound by Q_1/Q_2. The definition of absorption by Clark and Keenan that I have referred to goes as follows: for every $B, C \subseteq A$ and $D \subseteq A^2$,

$(Q_1/Q_2)_A(B, C, D) = \mathrm{T}$ iff $(Q_1)_A(\{a \in A : a \in B\},$
$\{b \in A : (Q_2)_A(\{b \in A : b \in C\}, \{b \in A : \langle a, b\rangle \in D\}) = \mathrm{T}\}) = \mathrm{T}$.

This definition is intended to "simulate" quantifications of the form

$(Q_1 x)[\Phi x, (Q_2 y)(\Psi y, \Xi xy)]$.

But as we have seen, it is not adequate for absorbing all well-formed formulas of the form

$(Q_1 x)[\Phi, (Q_2 y)(\Psi, \Xi)]$.

Note that the definition of satisfaction allows me to apply my absorbing quantifier whether x occurs free in Ψ or not. For example, I can apply absorption to

(95) Every man loves some woman,

or formally,

(96) $(\forall x)[Mx, (\exists y)(Wy, Lxy)]$,

and get

(97) $(\forall/\exists xy)(Mx, Wy, Lxy)$,

which has the right truth conditions. This is because the truth definition of (97) in a model $\mathfrak{A}$ is

$\mathfrak{A} \models (\forall/\exists xy)(Mx, Wy, Lxy)$ iff $\forall_A[\{a \in A : \mathfrak{A} \models Mx[g(x/a)]\},$
$\{a \in A : \exists_A(\{b \in A : \mathfrak{A} \models Wy[g(x|a)(y/b)]\},$
$\{b \in A : \mathfrak{A} \models Lxy[g(x/a), (y/b)]\}) = \mathrm{T}\}] = \mathrm{T}$,

and $\mathfrak{A} \models Wy[g(x/a), (y/b)]\}$ is equivalent to $\mathfrak{A} \models Wy[g(y/b)]\}$.

Absorption operators were originally investigated by Higginbotham and May (1981) in an attempt to account for the logical structure of cross reference, as in the Bach-Peters sentence

(98) Every pilot who shot at it hit some Mig that chased him.

May, in "Interpreting Logical Form" (1989), explains the issue as follows:

If scope is represented asymmetrically [as it is in formulas of form (91)], then the narrower scope quantifier cannot bind, as a bound variable, the pronoun contained within the broader scope phrase, which, in virtue of having broader scope, is outside its c-command domain. Thus if the *every*-phrase has broader scope, *it* cannot be a variable bound by the narrower *some*-phrase. Of course this problem disappears if the proper structure associated with [(98)] at LF is one of symmetric c-command, since then *it* would reside within the c-command domain of *some Mig that chased him* simultaneously with *him* residing within the c-command domain of *every pilot who shot at it*. [Absorption is then presented as] a structural readjustment of asymmetric structures into symmetric ones.[2]

I will not describe the exchange of views regarding this matter in the linguistic literature.[3] However, I would like to propose for consideration two formalizations of (98) in the spirit of May's suggestion.

First consider the 2-place predicative quantifier $\exists^{*1,1}$, which I will call "the conditional existential quantifier" or "the conditional *some*." Given a universe A, I define $\exists^*_A$ as follows: for any $B, C \subseteq A$,

$\exists^*_A(B, C) = \text{T}$ iff either $B = \varnothing$ or $B \cap C \neq \varnothing$.

In terms of cardinality t-functions (see chapter 2), $\exists^*_A$ is defined by the function $t_A^{\exists^*}$ such that for any $(\alpha, \beta, \gamma, \delta)$ in its domain,

$t_A^{\exists^*}(\alpha, \beta, \gamma, \delta) = \text{T}$ iff either $\beta = 0$ or $\alpha \neq 0$.

Figure 4.1 helps elucidate the relation between $\exists^*$ and $t_A^{\exists^*}$. Clearly, if Φ, Ψ are wffs,

(99) $(\exists^* x)(\Phi, \Psi)$

is logically equivalent to

(100) $(\exists x)\Phi \to (\exists x)(\Phi \,\&\, \Psi)$.

The quantifier $\exists^*$ might be used to interpret such English sentences as

(101) Every boy who chased a unicorn caught one,

understood as having the same truth conditions as

(102) $(\forall x)\{Bx \to [(\exists y)(Uy \,\&\, CHxy) \to (\exists y)(Uy \,\&\, CHxy \,\&\, Cxy)]\}$,

with the obvious symbolization key for B, U, CH and C. The formal

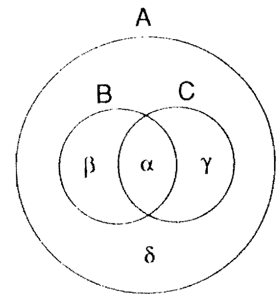

Figure 4.1

sentence (102) is equivalent to

(103) $(\forall x)[Bx \to (\exists^* y)(Uy \,\&\, CHxy, Cxy)]$,

which in some respects is closer in form to (101). Returning to the Bach-Peters sentence (98), the meaning of (98) seem to be captured by

(104) $(\forall x)\{Px \to [(\exists y)(My \,\&\, Cyx \,\&\, Sxy) \to (\exists y)(My \,\&\, Cyx \,\&\, Sxy \,\&\, Hxy)]\}$,

with the obvious readings for P, M, C, S, and H. (In understanding (98) as having the same meaning as (104), I follow Higginbotham and May in "Questions, Quantifiers, and Crossing" and Clark and Keenan in "The Absorption Operator and Universal Grammar."[4]) However, although (104) avoids the problem of cross binding, it does not appear to have the same logical structure as (98). I propose, therefore, that we assign to (98) the logical form

(105) $(\forall x)[Px \to (\exists^* y)(My \,\&\, Cyx \,\&\, Sxy, Hxy)]$.

Alternatively, we can analyze (98) as

(106) $(\forall x)[Px, (\exists^* y)(My \,\&\, Cyx \,\&\, Sxy, Hxy)]$,

which is obtained from (105) by replacing the 1-place $\forall$ by its 2-place variant. Both (105) and (106) are equivalent to (104), but I think they offer a better semantic representation of (98) than does (104), while solving the problem of cross binding just as well. If absorption is still desirable, we can apply it to the linear pair $\langle \forall, \exists^* \rangle$. We then obtain

(107) $(\forall/\exists^{*1,2,2}xy)(Px, My \,\&\, Cyx \,\&\, Sxy, Hxy)$.

Finally, to increase the structural similarity with (98), we can rewrite (107) using a quantifier equivalent to $\forall/\exists^{*1,2,2}$ but of the type $\langle 1, 2, 2, 2 \rangle$. This quantifier will be so defined that

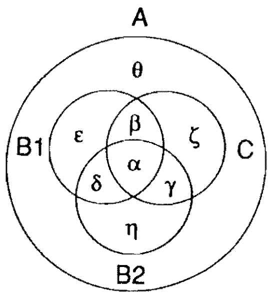

Figure 4.2

(108) $(\forall/\exists^{*1,2,2,2}xy)(Px, Sxy, Hxy, My \,\&\, Cyx)$

is equivalent to (107). Alternatively, we can construct a 3-place variant of $\exists^*$ and replace (106) with

(109) $(\forall x)[Px, (\exists^{*1,1,1}y)(Sxy, Hxy, My \,\&\, Cyx)]$.

The quantifier $\forall/\exists^{*1,2,2,2}$ will then be obtained by absorption from $\langle\forall, \exists^{*1,1,1}\rangle$ in the obvious way. Formally, there is no problem in constructing "superfluous" versions of quantifiers, and indeed, in chapter 2, I noted that such terms are common in natural languages. The 3-place $\exists^*$ is defined by a function t as follows:

$t_A(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta, \theta) = \mathrm{T}$ iff either $\delta = 0$ or $\alpha \neq 0$

The relation between $\exists^{*1,1}$ and $\exists^{*1,1,1}$ becomes clear when we compare figure 4.1 to figure 4.2. (Given an x, B1 represents "Sxy," B2 represents "$My \,\&\, Cyx$," and C represents "Hxy.")

If my analysis is correct, it is left for the linguist to account for the occurrence of "superfluous" logical forms in certain natural-language constructions. I will not attempt such an account. It may indeed be the case that what is superfluous from a purely logical point of view is significant from a linguistic viewpoint.

Pair quantifiers

Pair quantifiers are 1-place quantifiers satisfying Higginbotham and May's invariance condition (c) but not (b) or (a). Here are two examples:

(110) Three villagers and two townsmen exchanged blows.

(111) Two Germans and three Americans will challenge each other in the next tournament.

Note that the number words in each of these sentences can themselves be construed as quantifiers. But as predicative quantifiers, neither is within the scope of the other. Therefore, these are not ordinary predicative quantifications but fall under the category of branching quantifications. A general analysis of the branching structure will be given in chapter 5.

Other pair quantifiers express various correspondence relationships. Thus, treating modes of unhappiness as individuals (or allowing ascent to second-order logic), we can analyze Tolstoy's opening to *Anna Karenina* as a pair quantification stating a one-to-one correspondence:

(112) Each unhappy family is unhappy in its own way.

Other examples of pair quantifiers are

(113) Courses vary in the students they attract.

(114) My countrymen are divided in their views about war and peace.

(115) Different students answered different questions on the exam.[5]

Statements of the form "For every A there is a B," discussed by G. Boolos (1981), can also be construed as pair quantifications.

(116) For every drop of rain that falls, a flower grows.[6]

Sentences (112) to (116) include quantifiers that take into account not only cardinalities but more refined formal features of objects standing in relations. In particular, these quantifiers discern sameness and difference between objects within (though not across) each domain of a given relation. Thus the 1-place quantifier "vary," as in

(Vary xy)Rxy,

is defined, for each cardinal α, by a logical operator o_α such that, for example,

$o^{\text{vary}}_{10}([\{\langle 1, 6\rangle, \langle 2, 6\rangle, \langle 3, 6\rangle, \langle 4, 6\rangle, \langle 5, 7\rangle\}]) = \mathrm{F}$,

while

$o^{\text{vary}}_{10}([\{\langle 1, 6\rangle, \langle 2, 7\rangle, \langle 3, 8\rangle, \langle 4, 3\rangle, \langle 5, 9\rangle\}]) = \mathrm{T}$.

Finally, I would like to point out a construction with strong relational quantifiers that is more common in Hebrew than in English. Consider the following situation: A group of objects is divided into pairwise disjoint subgroups of n members each, and a certain condition is set on the members of each group. For example, given an initial group of students, the members of each subgroup are assigned a room in the dormitory, or given an initial group of soldiers (say an army in disarray), the members of each subgroup "fight their own war." These situations are described concisely in the sentences below.

(117) Every four students will receive a room.

Kol arba'ah studentim yekablu heder.

(118) Every several soldiers fought their own war.

Kol kama hayalim lahamu at milhamtam shelahem.

"Every four" and "every several" (in the sense indicated above) are naturally understood as strong relational quantifiers that distinguish partitions of a certain size in the domain of the quantified relation.

Strong relational quantifiers

Strong relational quantifiers are quantifiers satisfying the strongest invariance condition (d) in Higginbotham and May's list but not (a) through (c). As we have seen above, "couple" quantifiers fall under this category. Other genuinely strong relational quantifiers are quantifiers requiring the detection of sameness and difference across domains. Thus the quantifier "Reflexive *xy*" is a strong quantifier, as are all quantifiers attributing order properties to relations in their scope. Consider the following examples:

(119) Parenthood is an antireflexive relation.

(120) Forty workers elected a representative from among themselves.

We have completed the description of first-order Unrestricted Logic (UL) based on the philosophical conception developed in chapter 3. This conception was formally and linguistically elaborated in the present chapter. Along with Lindström's original semantics, I have proposed a "constructive" method for representing logical terms with ordinal functions. This method constitutes a natural extension of Mostowski's work on 1-place predicative quantifiers. Some philosophical issues concerning the new conception of logic will be discussed in chapter 6. But first I would like to investigate the impact of the generalization of quantifiers on another new logical theory. This theory has to do not with logical particles but with complex structures of logical particles. It is the theory of *branching quantification.*

Chapter 5
Ways of Branching Quantifiers

1 Introduction

Branching quantifiers were first introduced by L. Henkin in his 1959 paper "Some Remarks on Infinitely Long Formulas." By "branching quantifiers" Henkin meant a new, nonlinearly structured quantifier prefix whose discovery was triggered by the problem of interpreting infinitistic formulas of a certain form.[1] The branching (or partially ordered) quantifier prefix is, however, not essentially infinitistic, and the issues it raises have largely been discussed in the literature in the context of finitistic logic, as they will be here.

We would eventually like to know whether branching quantification is a genuine logical form. But today we find ourselves in an interesting situation where it is not altogether clear what the branching structure is. While Henkin's work purportedly settled the issue in the context of standard quantifiers, Barwise's introduction of new quantifiers into branching theory reopened the question. What happens when you take a collection of quantifiers, order them in an arbitrary partial ordering, and attach the result to a given formula? What truth conditions are to be associated with the resulting expression? Are these conditions compositionally based on the single quantifiers involved? Although important steps toward answering these questions were made by Barwise, Westerståhl, van Benthem, and others, the question is to my mind still open. Following the historical development, I will begin with standard quantifiers.

Initially there were two natural ways to approach branching quantification: as a generalization of the ordering of standard quantifier prefixes and as a generalization of Skolem normal forms.

A generalization of the ordering of standard quantifier prefixes

In standard modern logic, quantifier prefixes are linearly ordered, both syntactically and semantically. The syntactic ordering of a quantifier prefix $\langle(Q_1x_1), \ldots, (Q_nx_n)\rangle$ (where Q_i is either $\forall$ or $\exists$ for $1 \leq i \leq n$) mirrors the sequence of steps used to construct well-formed formulas with that quantifier prefix. Thus, if

(1) $(Q_1x_1) \ldots (Q_nx_n)\Phi(x_1, \ldots, x_n)$

is a well-formed formula, of any two quantifiers Q_ix_i and Q_jx_j ($1 \leq i \neq j \leq n$), the innermore precedes the outermore in the syntactic construction of (1). The semantic ordering of a quantifier prefix is the order of determining the truth (satisfaction) conditions of formulas with that prefix, and it is the backward image of the syntactic ordering. The truth of a sentence of the form (1) in a model $\mathfrak{A}$ with universe A is determined in the following order of stages:[2]

1. Conditions of truth (in $\mathfrak{A}$) for $(Q_1x_1)\Psi_1(x_1)$, where
   $\Psi_1 = (Q_2x_2) \ldots (Q_nx_n)\Phi(x_1, x_2, \ldots, x_n)$
2. Conditions of truth for $(Q_2x_2)\Psi_2(x_2)$, where
   $\Psi_2 = (Q_3x_3) \ldots (Q_nx_n)\Phi(a_1, x_2, x_3, \ldots, x_n)$
   and a_1 is an arbitrary element of A

   $\vdots$

n. Conditions of truth for $(Q_nx_n)\Psi_n(x_n)$, where
   $\Psi_n = \Phi(a_1, a_2, \ldots, a_{n-1}, x_n)$
   and $a_1, \ldots, a_{n-1}$ are arbitrary elements of A

We obtain branched quantification by relaxing the requirement that quantifier prefixes be *linearly* ordered and allowing *partial* ordering instead. It is clear what renouncing the requirement of linearity means syntactically. But what does it mean semantically? What would a partially ordered definition of truth for multiply quantified sentences look like? Approaching branching quantifiers as a generalization on the ordering of quantifiers in standard logic leaves the issue of their correct semantic definition an open question.

A generalization of Skolem normal forms

The Skolem normal form theorem says that every first-order formula is logically equivalent to a second-order prenex formula of the form

(2) $(\exists f_1) \ldots (\exists f_m)(\forall x_1) \ldots (\forall x_n)\Phi$,

where $x_1, \ldots, x_n$ are individual variables, $f_1, \ldots, f_m$ are functional variables ($m, n \geq 0$), and Φ is a quantifier-free formula.[3] This second-order formula is a *Skolem normal form*, and the functions satisfying a Skolem normal form are *Skolem functions*.

The idea is roughly that given a formula with an individual existential quantifier in the scope of one or more individual universal quantifiers, we obtain its Skolem normal form by replacing the former with a functional existential quantifier governing the latter. For example,

(3) $(\forall x)(\forall y)(\exists z)\Phi(x, y, z)$

is equivalent to

(4) $(\exists f^2)(\forall x)(\forall y)\Phi[x, y, f^2(x, y)]$.

The functional variable f^2 in (4) replaces the individual variable z bound by the existential quantifier $(\exists z)$ in (3), and the arguments of f^2 are all the individual variables bound by the universal quantifiers governing $(\exists z)$ there. It is characteristic of a Skolem normal form of a first-order formula with more than one existential quantifier that for any two functional variables in it, the set of arguments of one is included in the set of arguments of the other. Consider, for instance, the Skolem normal form of

(5) $(\forall x)(\exists y)(\forall z)(\exists w)\Phi(x, y, z, w)$,

namely,

(6) $(\exists f^1)(\exists g^2)(\forall x)(\forall z)\Phi[x, f^1(x), z, g^2(x, z)]$.

In general, Skolem normal forms of first-order formulas are formulas of the form (2) satisfying the following property:

(7) The functional existential quantifiers $(\exists f_1), \ldots, (\exists f_m)$ can be ordered in such a way that for all $1 \leq i, j \leq m$, if $(\exists f_j)$ syntactically precedes $(\exists f_i)$, then the set of arguments of f_i in Φ is essentially included in the set of arguments of f_j in Φ.[4]

This property reflects what W. J. Walkoe calls the "essential order" of linear quantifier prefixes.[5]

The existence of Skolem normal forms for all first-order formulas is thought to reveal a systematic connection between Skolem functions and existential individual quantifiers. However, this connection is not symmetric. Not all formulas of the form (2), general Skolem forms, are expressible in standard (i.e., linear) first-order logic. General Skolem forms not satisfying (7) are not.

It is natural to generalize the connection between Skolem functions and existential quantifiers into a complete correspondence. But such a generalization requires that first-order quantifier prefixes not be in general linearly ordered. The simplest Skolem form not satisfying (7) is

(8) $(\exists f^1)(\exists g^1)(\forall x)(\forall z)\Phi[x, f^1(x), z, g^1(z)]$.

Relaxing the requirement of syntactic linearity, we can construct a "first-order" correlate for (8), namely

$$(9)\quad \begin{matrix}(\forall x)(\exists y)\searrow \\ \\ (\forall z)(\exists w)\nearrow\end{matrix}\Phi(x, y, z, w).$$

We see that the semantic structure of a partially ordered quantifier prefix is introduced in this approach *together* with (or even *prior* to) the syntactic structure. The interpretation of a first-order branching formula is fixed to begin with by its postulated equivalence to a second-order, linear Skolem form.

Do the two generalizations above necessarily coincide? Do second-order Skolem forms provide the only reasonable semantic interpretation for the syntax of partially ordered quantified formulas? The definition of branching quantifiers by generalized Skolem functions was propounded by Henkin, who recommended it as "natural." Most subsequent writers on the subject took Henkin's definition as given. I was led to reflect on the possibility of alternative definitions by J. Barwise's paper "On Branching Quantifiers in English" (1979). Barwise shifted the discussion from standard to generalized branching quantifiers, forcing us to rethink the principles underlying the branching structure. Reviewing the earlier controversy around Hintikka's purported discovery of branching quantifier constructions in natural language and following my own earlier inquiry into the nature of quantifiers, I came to think that both logico-philosophical and linguistic considerations suggest further investigation of the branching form.

2 Linguistic Motivation

In "Quantifiers vs. Quantification Theory" (1973), J. Hintikka first pointed out that some quantifier constructions in English are branching rather than linear. A well-known example is,

(10) Some relative of each villager and some relative of each townsman hate each other.[6]

Hintikka says, "This [example] may … offer a glimpse of the ways in which branched quantification is expressed in English. Quantifiers occurring in conjoint constituents frequently enjoy independence of each other, it seems, because a sentence is naturally thought of as being symmetrical semantically vis-à-vis such constituents."[7] Another linguistic form of the branching-quantifier structure is illustrated by

(11) Some book by every author is referred to in some essay by every critic.[8]

Hintikka's point is that sentences such as (10) and (11) contain two *independent* pairs of iterated quantifiers, the quantifiers in each pair being outside the scope of the quantifiers in the other. A standard first-order formalization of such sentences—for instance, that of (10) as

(12) $(\forall x)(\exists y)(\forall z)(\exists w)(Vx \;\&\; Tz \rightarrow Ryx \;\&\; Rwz \;\&\; Hyw \;\&\; Hwy)$

or

(13) $(\forall x)(\forall z)(\exists y)(\exists w)(Vx \;\&\; Tz \rightarrow Ryx \;\&\; Rwz \;\&\; Hyw \;\&\; Hwy)$

(with the obvious readings for V, T, R, and H)—creates dependencies where none should exist. A branching-quantifier reading, on the other hand,

$$(14)\quad \begin{matrix}(\forall x)(\exists y)\searrow \\ \\ (\forall z)(\exists w)\nearrow\end{matrix} Vx \;\&\; Tz \rightarrow Ryx \;\&\; Rwz \;\&\; Hyw \;\&\; Hwy,$$

accurately simulates the dependencies and independencies involved. Hintikka does not ask what truth conditions should be assigned to (14) but rather assumes that it is interpreted in the "usual" way as

(15) $(\exists f^1)(\exists g^1)(\forall x)(\forall z)\{Vx \;\&\; Tz \rightarrow R(f^1(x), x) \;\&\; R(g^1(z), z) \;\&$
$H(f^1(x), g^1(z)) \;\&\; H(g^1(z), f^1(x))\}$.

Hintikk'a paper brought forth a lively exchange of opinions, and G. Fauconnier (1975) raised the following objection (which I formulate in my own words): (15) implies that the relation of mutual hatred between relatives of villagers and relatives of townsmen has what we might call a *massive nucleus*—one that contains at least one relative of each villager and one relative of each townsman—and such that *each* villager relative in the nucleus hates *all* the townsman relatives in it, and vice versa. However, Fauconnier objects, it is not true that every English sentence with syntactically independent quantifiers implies the existence of a *massive nucleus* of objects standing in the quantified relation. For instance,

(16) Some player of every football team is in love with some dancer of every ballet company

does not.[9] It is compatible with the assumption that men are in love with one woman at a time (and that dancers/football-players do not belong to more than one ballet-company/football-team at a time). Even if Hintikka's interpretation of (10) is correct, Fauconnier continues, i.e., even if (10) implies the existence of a *massive nucleus* of villagers and townsmen in mutual hatred, (16) *does not* imply the existence of a *massive nucleus* of football players in love with dancers. Hintikka's interpretation, therefore, is not appropriate to all scopewise independent quantifiers in natural language. I illustrate the issue graphically in figures 5.1 and 5.2. The point is accentuated in the following examples:

(17) Some player of every football team is *the* boyfriend of some dancer of every ballet company.

(18) Some relative of each villager and some relative of each townsman are married (to one another).

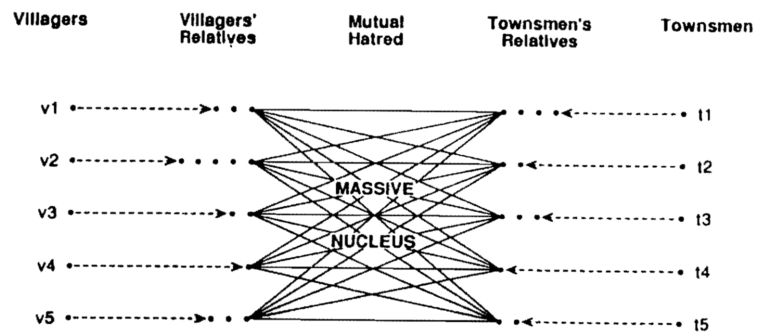

Figure 5.1

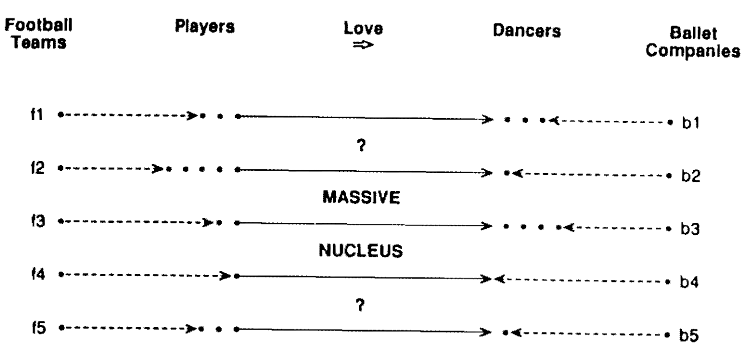

Figure 5.2

Is (17) logically false? Does (18) imply that the community in question is polygamous?

Fauconnier's conclusion is that natural-language constructions with quantifiers independent in scope are sometimes branching and sometimes linear, depending on the context. The correct interpretation of (16), for instance, is

(19) $(\forall x)(\forall y)(\exists z)(\exists w)(Fx \;\&\; By \rightarrow Pzx \;\&\; Dwy \;\&\; Lzw)$.

Thus, according to Fauconnier, the only alternative to "massive nuclei" is linear quantification.

We can, however, approach the matter somewhat differently. Acknowledging the semantic independence of syntactically unnested quantifiers in general, we can ask, Why should the independence of quantifiers have anything to do with the existence of a "massive nucleus" of objects standing in the quantified relation? Interpreting branching quantifiers nonlinearly, yet without commitment to a "massive nucleus," would do justice both to Hintikka's insight regarding the nature of scope-independent quantifiers and to Fauconnier's (and others') observations regarding the multiplicity of situations that such quantifiers can be used to describe. We are thus led to search for an alternative to Henkin's definition that would avoid the problematical commitment.

3 Logico-philosophical Motivation

Why are quantifier prefixes in modern symbolic logic linearly ordered? M. Dummett (1973) ascribes this feature of quantification theory to the genius of Frege. Traditional logic failed because it could not account for the validity of inferences involving multiple quantification. Frege saw that the problem could be solved if we construed multiply quantified sentences as complex *step-by-step* constructions, built by repeated applications of the simple logical operations of universal and/or existential quantification. This step-by-step syntactic analysis of multiply quantified sentences was to serve as a basis for a corresponding step-by step semantic analysis that unfolds the truth conditions of one constructional stage, i.e., a *singly quantified formula*, at a time. (See section 1 above.) In other words, by Frege's method of logical analysis the problem of defining truth for a quantified many-place relation was reduced to that of defining truth for a series of quantified predicates (1-place relations), a problem whose solution was essentially known.[10] The possibility of such a reduction was based, however, on a particular way of representing relations. In Tarskian semantics this form of representation is reflected in the way in which the linear steps in the definition of truth are "glued" together, namely by a relative expression synonymous with "for each one of which" ("f.e.w."). Thus, for example, the Fregean-Tarskian definition of truth for

(20) $(Q_1 x)(Q_2 y)(Q_3 z)R^3(x, y, z)$,

where Q_1, Q_2, and Q_3 are either $\forall$ or $\exists$, proceeds as follows: (20) is true in a model $\mathfrak{A}$ with a universe A *iff* there are $q_1 a$'s in A, f.e.w. there are $q_2 b$'s in A, f.e.w. there are q_3 c's in A such that "$R^3(a, b, c)$" is true in $\mathfrak{A}$, where q_1, q_2, and q_3 are the quantifier conditions associated with Q_1, Q_2, and Q_3 respectively.[11]

Intuitively, the view of R^3 embedded in the definition of truth for (20) is that of a *multiple tree*. (See figure 5.3.) Each row in the multiple tree represents one domain of R^3 (the extension of one argument place of R^3); each tree represents the restriction of R^3 to some one element of the domain listed in the upper row. In this way the extension of the second domain is represented *relative* to that of the first, and the extension of the third relative to the (already relative) representation of the second. Different quantifier prefixes allow different multiple-tree views of relations, but Frege's linear quantification limits the expressive power of quantifier prefixes to properties of relations that are discernible in a mutliple-tree representation.

We can describe the sense in which (all but the outermost) quantifiers in a linear prefix are semantically dependent as follows: a *linearly dependent* quantifier assigns a property not to a complete domain of the relation quantified but to a domain relativized to individual elements of another domain higher up in the multiple tree. It is characteristic of a linear quantifier prefix that each quantifier (but the outermost) is directly dependent on exactly one other quantifier. I will therefore call linear quantifiers *unidependent-* or *simply dependent*.

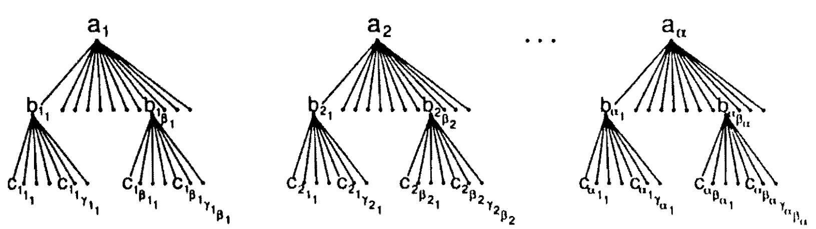

Figure 5.3

There are two natural alternatives to simple dependence: (1) no dependence, i.e., *independence*, and (2) *complex dependence*. These correspond to two ways in which we can view relations in a nonlinear manner: we can view each domain separately as complete and unrelativised, or we can view a whole cluster of domains at once in their mutual relationships.

Syntactically, I will represent an *independent* quantification by

$$(21)\quad \left.\begin{array}{c}(Q_1 x_1)\\ \vdots\\ (Q_n x_n)\end{array}\right|\; R^n(x_1, \ldots, x_n)$$

and a *complex* quantification by

$$(22)\quad \left.\begin{array}{c}(Q_1 x_1)\\ \vdots\\ (Q_n x_n)\end{array}\right\rangle\!\!\Rightarrow R^n(x_1, \ldots, x_n).$$

Of course, there are many complex patterns of dependence among quantifiers. These can be represented by various partially ordered prefixes.

Our analysis indicates that the concept of independent quantification is different from that of complex quantification. Therefore, the first question regarding the correct interpretation of natural-language sentences with branching quantifiers is, Are the quantifiers in these sentences independent or complex?

4 Independent Branching Quantifiers

It is easy to give a precise definition of *independent quantification*:

$$(23)\quad \left.\begin{array}{c}(Q_1 x)\\ \\ (Q_2 y)\end{array}\right|\; \Phi(x, y) =_{\mathrm{df}} (Q_1 x)(\exists y)\Phi(x, y) \;\&\; (Q_2 y)(\exists x)\Phi(x, y),$$

or more generally,

$$(24)\quad \left.\begin{array}{c}(Q_1 x_1)\\ \vdots\\ (Q_n x_n)\end{array}\right|\; \Phi(x_1, \ldots, x_n) =_{\mathrm{df}}$$

$$(Q_1 x_1)(\exists x_2)\ldots(\exists x_n)\Phi(x_1, \ldots, x_n) \;\&\; \ldots \;\&$$
$$(Q_n x_n)(\exists x_1)\ldots(\exists x_{n-1})\Phi(x_1, \ldots, x_n).$$

This new definition of nonlinear quantification is very different from that of Henkin's. Independent quantification is essentially first-order. It does not involve commitment to a "massive nucleus" or to any other complex structure of objects standing in the quantified relation. Therefore, it enables us to analyze natural-language sentences with scope-independent quantifiers in a straightforward manner and without forcing any independent quantifier into a nested position. I thus propose (23) as a definition of branching quantifiers as *independent* quantifiers. Linguistically, this construal is supported by the fact that "and" often appears as a "quantifier connective" in natural-language branching structures in a way which might indicate a shift from its "original" position as a sentential connective. Moreover, natural-language branching quantifiers are symmetrical in much the same way that the conjuncts in my definition are. An English sentence with standard quantifiers that appears to exemplify independent quantification is

(25) Nobody loves nobody,

understood as "Nobody loves anybody."[12] I will symbolize (25) as

$$(26)\quad \begin{array}{l|l} (\sim\exists x) & \\ & Lxy \\ (\sim\exists y) & \end{array}$$

and interpret it as

(27) $\sim(\exists x)(\exists y)Lxy \;\&\; \sim(\exists y)(\exists x)Lxy.$

By extending our logical vocabulary to 1-place Mostowskian quantifiers, we will be able to interpret the following English sentences as independent branching quantifications:

(28) Three elephants were chased by a dozen hunters.

(29) Four Martians and five Humans exchanged insults.

(30) An odd number of patients occupied an even number of beds.

The "independent" interpretation of (28) to (30) reflects a "cumulative" reading, under which no massive nucleus, or any other complex relationship between the domain and the range of the relation in question, is intended.[13] We thus understand (28) as saying that the relation "elephant x was chased by hunter y" includes three individuals in its domain and a dozen individuals in its range. And this reading is captured by (23). Similarly, (23) yields the cumulative interpretations of (29) and (30).

The extension of the definition to 2-place Mostowskian quantifiers (which in this chapter I symbolize as Q^2 rather than $Q^{1,1}$) will yield independent quantifications of the form

$$(31)\quad \begin{array}{l|ll} (Q^2_1 x) & \Psi_1 x, & \\ & & \Phi xy. \\ (Q^2_2 y) & \Psi_2 y, & \end{array}$$

Here, however, we can apply the notion of independent quantification in several ways. Given a binary relation R, two sets A and B, and two quantifier conditions q_1 and q_2, we can say the following:

a. The relation R has q_1 As in its domain and q_2 Bs in its range.
b. The relation $A \upharpoonleft R \upharpoonright B$ has q_1 elements in its domain and q_2 elements in its range (where $A \upharpoonleft R \upharpoonright B$ is obtained from R by restricting its domain to A and its range to B).
c. The relation $A \upharpoonleft R \upharpoonright B$ has q_1 As in its domain and q_2 Bs in its range.
d. The relation $R \upharpoonright B$ has q_1 As in its domain and q_2 Bs in its range.

⋮

It is easy to see that (a) through (d) are not equivalent.[14] However, for the examples discussed here it suffices to define (31) for case (c). I thus propose as the definition of a pair of 2-place independent quantifiers

$$(32)\quad \begin{array}{l|ll} (Q^2_1 x) & \Psi_1 x, & \\ & & \Phi xy =_{df} (Q^2_1 x)[\Psi_1 x, (\exists y)(\Psi_1 x \;\&\; \Psi_2 y \;\&\; \Phi xy)] \;\& \\ (Q^2_2 y) & \Psi_2 y, & \qquad\qquad (Q^2_2 y)[\Psi_2 y, (\exists x)(\Psi_1 x \;\&\; \Psi_2 y \;\&\; \Phi xy)]. \end{array}$$

When Q^2_1 and Q^2_2 satisfy the property of *living on*, i.e., when $(Q^2 x)(\Phi x, \Psi x)$ is logically equivalent to $(Q^2 x)(\Phi x, \Phi x \;\&\; \Psi x)$, we can replace (32) with the simpler

$$(33)\quad \begin{array}{l|ll} (Q^2_1 x) & \Psi_1 x, & \\ & & \Phi xy =_{df} (Q^2_1 x)[\Psi_1 x, (\exists y)(\Psi_2 y \;\&\; \Phi xy)] \;\& \\ (Q^2_2 y) & \Psi_2 y, & \qquad\qquad (Q^2_2 y)[\Psi_2 y, (\exists x)(\Psi_1 x \;\&\; \Phi xy)]. \end{array}$$

Using this definition, we can interpret (34) and (35) below as independent quantifications:

(34) All the boys ate all the apples.[15]

(35) Two boys ate half the apples.

We can also analyze (28) to (30) as independent quantifications of the form (33).[16]

What about Hintikka's (10) and Fauconnier's (16)? Should we interpret these as independent branching quantifications of the form (33)? Under such an interpretation, (10) would say that the relation of mutual hatred between relatives of villagers and relatives of townsmen includes at least one relative of each villager in its domain and at least one relative of each townsman in its range; (16) would be understood as saying that the relation of love between football players and ballet dancers includes at least one player of each football team in its domain and at least one dancer from each ballet company in its range. Such interpretations would be compatible with both figures 5.1 and 5.2. Later on I will suggest a test to determine whether the intended interpretation of a given natural-language sentence with branching quantifiers is that of an independent or complex quantification, and this might give us a clue regarding Hintikka's and Fauconnier's sentences. As for the linear option, here the question is whether one pair of quantifiers is within the scope of the other. Generally, I would say that when "and" appears as a quantifier connective, that is, "Q_1 *As and* Q_2 *Bs* stand in relation *R*," the quantification is not linear. However, when the quantification is of the form "Q_1 *As R* Q_2 *Bs*," the situation is less clear. (For further discussion, see May 1989 and van Benthem 1989.[17]) I should note that sometimes the method of semantic representation itself favors one interpretation over another. For example, in standard semantics, relations are so represented that it is impossible for the range of a given binary relation to be empty when its domain is not empty. Thus a quantification of the form "Three *As* stand in the relation *R* to zero *Bs*" would be logically false if interpreted as independent branching quantification. To render it logically contingent, we may construe it as a nested quantification of two 1-place predicative quantifiers, and this gives us the linear reading.

5 Barwise's Generalization of Henkin's Quantifiers

I now turn to *complex quantification*. Evidently, Henkin's quantifiers belong in this category. I ask: What kind of information on a quantified relation does a complex quantifier prefix give us? As we shall soon see, the shift to a more general system of quantifiers, namely Mostowski's 1- and 2-place predicative quantifiers, throws a new light on the nature of complex branching quantification.

Barwise (1979) generalized Henkin's definition of standard branching quantifiers to 1-place monotone-increasing Mostowskian quantifiers in the following way:[18]

(36) $$\begin{array}{l} (Q_1 x) \\ \qquad > \Phi xy =_{df} (\exists X)(\exists Y)[(Q_1 x)Xx \,\&\, (Q_2 y)Yy \,\& \\ (Q_2 y) \qquad\qquad\qquad (\forall x)(\forall y)(Xx \,\&\, Yy \rightarrow \Phi xy)]. \end{array}$$ [19]

Technically, the generalization is based on a relational reading of the Skolem functions in Henkin's definition. Thus, Barwise's equivalent of Henkin's (8) is

(37) $$(\exists R)(\exists S)[(\forall x)(\exists y)Rxy \,\&\, (\forall z)(\exists w)Szw \,\& \\ (\forall x)(\forall y)(\forall z)(\forall w)(Rxy \,\&\, Szw \rightarrow \Phi(x, y, z, w))].$$

Clearly, Barwise's quantifiers, like Henkin's, are *complex*, not independent, branching quantifiers.

Barwise suggested that this generalization enables us to give English sentences with unnested monotone-increasing generalized quantifiers a "Henkinian" interpretation similar to Hintikka's interpretation of (10) and (11). Here are two of his examples:[20]

(38) Most philosophers and most linguists agree with each other about branching quantification.

(39) Quite a few boys in my class and most girls in your class have all dated each other.

To interpret (38) and (39), we have to extend (36) to 2-place predicative quantifiers. This we do as follows: Let Q_1^2 and Q_2^2 be 2-place monotone-increasing predicative quantifiers. Then

(40) $$\begin{array}{l} (Q_1^2 x) \cdot \Psi_1 x, \\ \qquad \cdot \qquad \Phi xy =_{df} \\ (Q_2^2 y) \cdot \Psi_2 y, \end{array}$$
$$(\exists X)(\exists Y)[(Q_1^2 x)(\Psi_1 x, Xx) \,\&\, (Q_2^2 y)(\Psi_2 y, Yy) \,\& \\ (\forall x)(\forall y)(Xx \,\&\, Yy \rightarrow \Phi xy)].$$

We can now interpret (38) as

(41) $$\begin{array}{l} (M^2 x) \cdot Px, \\ \qquad \cdot \qquad Axy \,\&\, Ayx =_{df} \\ (M^2 y) \cdot Ly, \end{array}$$
$$(\exists X)(\exists Y)[(M^2 x)(Px, Xx) \,\&\, (M^2 y)(Ly, Yy) \,\& \\ (\forall x)(\forall y)(Xx \,\&\, Yy \rightarrow Axy \,\&\, Ayx)],$$

with the obvious readings of *P*, *L*, *A* and where "M^2" stand for the 2-place "most." We interpret (39) in a similar manner.

Barwise emphasized that his definition of branching monotone-increasing generalized quantifiers is not applicable to monotone-decreasing, non-monotone, or mixed branching quantifiers.[21] This is easily explained by the absurd results of applying (36) to such quantifiers: (36) would render any monotone-decreasing branching formula vacuously true (by taking *X* and *Y* to be the empty set); it would render false non-monotone branching formulas true, as in the case of "Exactly one *x* and exactly one *y* stand in the relation *R*," where *R* is universal and the cardinality of the universe is larger than 1.

Barwise proposed the following definition for a pair of 1-place monotone-decreasing branching quantifiers:

$$(42)\quad \left.\begin{array}{l}(Q_1x)\\ \\ (Q_2y)\end{array}\right> \Phi xy =_{df} (\exists X)(\exists Y)[(Q_1x)Xx \mathbin{\&} (Q_2y)Yy \mathbin{\&} (\forall x)(\forall y)(\Phi xy \to Xx \mathbin{\&} Yy)].$$ [22]

Definition (42), or its counterpart for 2-place quantifiers, provides an intuitively correct semantics for English sentences with a pair of unnested monotone-decreasing quantifiers. Consider, for instance,

(43) Few philosophers and few linguists agree with each other about branching quantification.

As to non-monotone and mixed branching quantifiers, Barwise left the former unattended and skeptically remarked about the latter, "There is no sensible way to interpret

$$(s)\quad \left.\begin{array}{l}Q_1x\\ \\ Q_2y\end{array}\right> A(x, y)$$

when one [quantifier] is increasing and the other is decreasing. Thus, for example,

(t) ?Few of the boys in my class and most of the girls in your class have all dated each other.

appears grammatical, but it makes no sense."[23]

Barwise's work suggests that the semantics of branching quantifiers depends on the monotonic properties of the quantifiers involved. The truth conditions for a sentence with branching monotone-increasing quantifiers are altogether different from the truth conditions for a sentence with branching monotone-decreasing quantifiers, and truth for sentences with mixed branching quantifiers is simply undefinable. Is the meaning of branching quantification as intimately connected with monotonicity as Barwise's analysis may lead one to conclude?

First, I would like to observe that Barwise interprets branching monotone-decreasing quantifiers simply as *independent* quantifiers: when Q_1 and Q_2 are monotone-decreasing (42) is logically equivalent to my (23). The latter definition, as we have seen, has meaning—the same meaning—for all quantifiers, irrespective of monotonicity. On this first-order reading, (43) says that the relation of mutual agreement about branching quantification between philosophers and linguists includes (at most) few philosophers in its domain and (at most) few linguists in its range.

Barwise explained the limited applicability of (36) in the following way: Every formula of the form

(44) $(Qx)\Phi x$,

where Q is monotone-increasing, is logically equivalent to a second-order formula of the form

(45) $(\exists X)[(Qx)Xx \mathbin{\&} (\forall x)(Xx \to \Phi x)]$,

which is structurally similar to (36). This fact establishes (36) as the correct definition of branching monotone-increasing quantifiers. However, (45) is not a second-order representation of quantified formulas with non-monotone-increasing quantifiers. Hence (36) does not apply to branching quantifiers of the latter kind. The definition of branching monotone-decreasing quantifiers by (42) is explained in a similar manner: when Q is monotone-decreasing, (44) is logically equivalent to

(46) $(\exists X)[(Qx)Xx \mathbin{\&} (\forall x)(\Phi x \to Xx)]$,

which is structurally similar to (42).[24]

I do not find this explanation convincing. Linear quantifiers vary with respect to monotonicity as much as branching quantifiers do, yet the semantic definition of linear quantifiers is the same for *all* quantifiers, irrespective of monotonicity. Linear quantification is also meaningful for all *combinations* of quantifiers. Why should the meaningfulness of the branching form stop short at mixed monotone quantifiers? Moreover, if the second-order representation of "simple" first-order quantifications determines the correct analysis of branching quantifications, Barwise has not shown that there is no second-order representation of (44) that applies *universally*, without regard to monotonicity.

6 A General Definition of Complex, Henkin-Barwise Branching Quantifiers

The conception of complex branching quantification embedded in Barwise's (36) assigns the following truth conditions to branching formulas of the form

$$(47)\quad \left.\begin{array}{l}(Q_1x)\\ \\ (Q_2y)\end{array}\right> \Phi xy,$$

where Q_1 and Q_2 are monotone-increasing:

DEFINITION 1 The branching formula (47) is true in a model $\mathfrak{A}$ with universe A iff there is at least one pair, $\langle X, Y\rangle$, of subsets of A for which the following conditions hold:

1. X satisfies the quantifier condition Q_1.
2. Y satisfies the quantifier condition Q_2.
3. *Each* element of X stands in the relation $\Phi^{\mathfrak{A}}$ to *all* the elements of Y.

The condition expressed by (3) I shall call the *each-all* (or *all-all*) condition on $\langle X, Y\rangle$ with respect to $\Phi^{\mathfrak{A}}$. We can then express definition 1 more succinctly as follows:

DEFINITION 2 The branching formula (47) is true in a model $\mathfrak{A}$ with a universe A iff there is at least one pair of subsets of the universe satisfying the *each-all* condition with respect to $\Phi^{\mathfrak{A}}$, with its first element satisfying Q_1 and its second element satisfying Q_2.

Set-theoretically, definition 2 says that $\Phi^{\mathfrak{A}}$ includes at least one *Cartesian product* of two subsets of the universe satisfying Q_1 and Q_2 respectively. (The "massive nucleus" of section 2 above was an informal term for a Cartesian product.)

Is the complex quantifier condition expressed by definition 2 meaningful only with respect to monotone-increasing quantifiers? I think that the idea behind this condition makes sense no matter what quantifiers Q_1 and Q_2 are. However, this idea is not adequately formulated in definition 2 as it now stands, since this definition fails to capture the intended condition when Q_1 and/or Q_2 are not monotone-increasing. In that case Q_1 and/or Q_2 set a limit on the size of sets X and/or Y such that $\langle X, Y\rangle$ satisfies the *each-all* condition with respect to $\Phi^{\mathfrak{A}}$: (47) is true only if a Cartesian product small enough or of a particular size is included in $\Phi^{\mathfrak{A}}$. But definition 2 in its present form cannot express this condition: if $\Phi^{\mathfrak{A}}$ includes a Cartesian product larger than required, definition 2 is automatically satisfied. This is because for any two nonempty sets A and B, if $A \times B$ is a Cartesian product included in $\Phi^{\mathfrak{A}}$, so is $A' \times B'$, where A' and B' are any proper subsets of A and B respectively. The difficulty, however, appears to be purely technical. We can overcome it by demanding that the condition be met by a *maximal*, not a sub-, Cartesian product. In other words, only maximal Cartesian products included in $\Phi^{\mathfrak{A}}$ should count as satisfying the *each-all* condition.

I thus add a *maximality condition* to definition 1 and arrive at the following general definition, in which no restrictions are set on Q_1 and Q_2:

DEFINITION 3 The branching formula (47) is true in a model $\mathfrak{A}$ with universe A iff there is at least one pair $\langle X, Y\rangle$ of subsets of A for which the following conditions hold:

1. X satisfies quantifier condition Q_1.
2. Y satisfies quantifier condition Q_2.
3. Each element of X stands in the relation $\Phi^{\mathfrak{A}}$ to all the elements of Y.
4. The pair $\langle X, Y\rangle$ is a maximal pair satisfying (3).

Referring to (3) and (4) as "the *maximal each-all* condition on $\langle X, Y\rangle$ with respect to Φ," we can reformulate definition 3 more concisely as follows:

DEFINITION 4 The branching formula (47) is true in a model $\mathfrak{A}$ with universe A iff there is at least one pair of subsets in the universe satisfying the *maximal each-all* condition with respect to $\Phi^{\mathfrak{A}}$ such that its first element satisfies Q_1 and its second element satisfies Q_2.

I thus propose to replace (36) with

(48) $$\begin{matrix}(Q_1x)\\(Q_2y)\end{matrix}\!\!>\Phi xy =_{df}$$
$$(\exists X)(\exists Y)\{(Q_1x)Xx \,\&\, (Q_2y)Yy \,\&\, (\forall x)(\forall y)(Xx \,\&\, Yy \to \Phi xy) \,\&\, (\forall X')(\forall Y')[(\forall x)(\forall y)((Xx \,\&\, Yy \to X'x \,\&\, Y'y) \,\&\, (X'x \,\&\, Y'y \to \Phi xy)) \to (\forall x)(\forall y)(Xx \,\&\, Yy \leftrightarrow X'x \,\&\, Y'y)]\}$$

as the definition of Henkin-Barwise complex branching quantifiers. We can rewrite (48) more succinctly, using common conventions, as

(49) $$\begin{matrix}(Q_1x)\\(Q_2y)\end{matrix}\!\!>\Phi xy =_{df}$$
$$(\exists X)(\exists Y)[(Q_1x)Xx \,\&\, (Q_2y)Yy \,\&\, X \times Y \subseteq \Phi \,\&\, (\forall X')(\forall Y')(X \times Y \subseteq X' \times Y' \subseteq \Phi \to X \times Y = X' \times Y')].$$

More concisely yet, we have

(50) $$\begin{matrix}(Q_1x)\\(Q_2y)\end{matrix}\!\!>\Phi xy =_{df}$$
$$(\exists X)(\exists Y)[(Q_1x)Xx \,\&\, (Q_2y)Yy \,\&\, (\forall X')(\forall Y')(X \times Y \subseteq X' \times Y' \subseteq \Phi \leftrightarrow X \times Y = X' \times Y')].$$

It is easy to see that whenever Q_1 and Q_2 are monotone-increasing, (49) is logically equivalent to (36). At the same time, (49) avoids the problems that arise when (36) is applied to non-monotone-increasing quantifiers.

Maximality conditions are very common in mathematics. Generally, when a structure is maximal, it is "complete" in some relevant sense.[25] The Henkin-Barwise branching quantifier prefix expresses a certain condition on sets (subsets of the quantified relation). And when we talk about sets, it is usually maximal sets that we are interested in. Indeed, conditions on sets are normally conditions on maximal sets. Consider, for instance, the statement "Three students passed the test." Would this statement be true had 10 students passed the test? But it would be if the quantifier "!3" set a condition on a nonmaximal set: a partial extension of "x is a student who passed the test" would satisfy that condition. Consider also "No student passed the test" and "Two people live in America."

The fact that quantification in general sets a condition on maximal sets (relations) is reflected by the equivalence of any first-order formula of the form

(44) $(Qx)\Phi x$,

no matter what quantifier Q is (monotone-increasing, monotone-decreasing or non-monotone), to

(51) $(\exists X)[(Qx)Xx \;\&\; X \subseteq \Phi \;\&\; (\forall X')(X \subseteq X' \subseteq \Phi \rightarrow X' = X)]$,

which expresses a maximality condition. The logical equivalence of (44) to (51) provides a further justification for the reformulation of (36) as (49).

We have seen that the two conceptions of nonlinear quantification discussed so far, *independence* (first-order) and *complex dependence* (second-order), have little to do with monotonicity or its direction. The two conceptions lead to entirely different definitions of the branching quantifier-prefix, both, however, universally applicable.

Linguistically, my suggestion is that to determine the truth conditions of natural-language sentences with a nonlinear quantifier-prefix, one has to ask not whether the quantifiers involved are monotone-increasing, monotone-decreasing, etc. but whether the prefix is independent or complex. My analysis points to the following clue: Complex Henkin-Barwise quantifications always include an inner *each-all* condition, explicit or implicit. Independent quantifications, on the other hand, do not include any such condition.

Barwise actually gave several examples of branching sentences with an explicit *each-all* condition:

(39) Quite a few boys in my class and most girls in your class have *all* dated each other.[26]

(52) Most of the dots and most of the stars are *all* connected by lines.[27]

Such an explicit "all" also appears in his

(t) Few of the boys in my class and most of the girls in your class have *all* dated each other.[28]

I therefore suggest that we interpret Barwise's (t) as an instance of (49).

Some natural examples of Henkin-Barwise complex branching quantifiers in English involve non-monotonic quantifiers. For example,

(53) A couple of boys in my class and a couple of girls in your class were *all* dating each other.

(54) An even number of dots and an odd number of stars are *all* connected by lines.

Another expression that seems to point to a complex branching structure (which indicates a second-order form) is "the same." Consider

(55) Most of my friends have applied to the *same* few graduate programs.

To interpret the above sentences accurately, we have to extend (49) to 2-place quantifiers. As in the case of 2-place independent quantifiers (see section 4 above), we can apply the notion of complex *each-all* quantification in more than one way. I will limit my attention to one of these, defining "Q_1 As and Q_2 Bs all stand in the relation R" as "There is at least one maximal Cartesian product included in $A \upharpoonleft R \upharpoonright B$ with Q_1 As in its domain and Q_2 Bs in its range." In symbols,

(56)
$$\begin{matrix} (Q_1^2 x) \cdot \Psi_1 x, \\ \cdot \\ (Q_2^2 y) \cdot \Psi_2 y, \end{matrix} \quad \Phi xy =_{df}$$

$$(\exists X)(\exists Y)[(Q_1^2 x)(\Psi_1 x, Xx) \;\&\; (Q_2^2 y)(\Psi_2 y, Yy) \;\&\; (\forall X')(\forall Y')(X \times Y \subseteq X' \times Y' \subseteq \Psi_1 \upharpoonleft \Phi \upharpoonright \Psi_2 \leftrightarrow X \times Y = X' \times Y')].$$

Linguistically, my account explains the meaning (function) of inner quantifiers that, like Barwise's "all," do not bind any new individual variables in addition to those bound by Q_1 and Q_2. A "standard" reading of such quantifiers is problematic, since all the variables are already bound by the outer quantifiers. On my analysis, these quantifiers point to a second-order condition.

Going back to the controversy regarding Hintikka's reading of natural-language sentences with symmetrical quantifiers, we can reformulate Fauconnier's criticism as follows: Some natural-language sentences with unnested quantifiers do not appear to contain, explicitly or implicitly, an inner *each-all* quantifier condition. On my analysis, these are not Henkin-Barwise branching quantifications. Whether Hintikka's (10) includes an implicit *each-all* condition, I leave an open question. (One way to justify Hintikka's claim that (10) is a Henkin sentence is to interpret "each" in "each other" as elliptic for "each-all.")

The reading of a natural-language branching quantification with no explicit *each-all* condition involves various linguistic considerations. Our logical point of view has so far indicated three possible readings: as an independent quantification, as a linear quantification, or as a Henkin-Barwise complex quantification. But as we will presently see, these are not the only options. In the next section I will introduce a "family of interpretations" that extends considerably the scope of nonlinear quantification.

7 Branching Quantifiers: A Family of Interpretations

The Henkin-Barwise definition of branching quantifiers, in its narrow as well as general form, includes two quantifier conditions in addition to those explicit in the definiendum: the outer quantifier condition "there is at least one pair $\langle X, Y\rangle$" and the inner (maximal) *each-all* quantifier condition. By generalizing these conditions, we arrive at a new definition schema whose instances comprise a family of semantic interpretations for multiple quantifiers. Among the members of this family are both the independent branching quantifiers of section 4 and the Henkin-Barwise complex quantifiers of section 6. This generalized definition schema delineates a totality of *forms of quantifier dependence*. Degenerate dependence is independence; linear dependence is a particular case of (nondegenerate) Henkin-Barwise dependence.[29]

We arrive at the definition schema in two steps. First we generalize the inner *each-all* quantifier condition (see definitions 1–4), and we obtain the following schema:

GENERALIZATION 1 A branching formula of the form (47) is true in a model $\mathfrak{A}$ with a universe A iff for at least one pair $\langle X, Y\rangle$ of subsets of the universe satisfying the maximal quantifier condition $\mathcal{Q}_1$ with respect to $\Phi^{\mathfrak{A}}$, X satisfies Q_1, and Y satisfies Q_2,

where $\mathcal{Q}_1$ represents any (first-order) maximal quantifier-condition on a pair of subsets of the universe with respect to $\Phi^{\mathfrak{A}}$. The following are a few instances of $\mathcal{Q}_1$:

Condition A: one–one The pair $\langle X, Y\rangle$ is a maximal pair such that each element of X stands in the relation $\Phi^{\mathfrak{A}}$ to exactly one element of Y and for each element of Y there is exactly one element of X that stands to it in the relation $\Phi^{\mathfrak{A}}$.

Condition B: each–two or more The pair $\langle X, Y\rangle$ is a maximal pair such that each element of X stands in the relation $\Phi^{\mathfrak{A}}$ to two or more elements of Y and for each element of Y there is an element of X that stands to it in the relation $\Phi^{\mathfrak{A}}$.

Condition C: each–more than ___ The pair $\langle X, Y\rangle$ is a maximal pair such that each element of X stands in the relation $\Phi^{\mathfrak{A}}$ to more than ___ elements of Y and for each element of Y there is an element of X that stands to it in the relation $\Phi^{\mathfrak{A}}$.

Condition D: each–at least half/at least half–each The pair $\langle X, Y\rangle$ is a maximal pair such that each element of X stands in the relation $\Phi^{\mathfrak{A}}$ to at least half the elements of Y and to each element of Y at least half the elements of X stand in the relation $\Phi^{\mathfrak{A}}$.

We can find natural-language sentences that exemplify generalization 1 by substituting conditions A through D for $\mathcal{Q}_1$:

(57) Most of my right-hand gloves and most of my left-hand gloves match (one to one).

(58) Most of my friends saw at least two of the same few Truffaut movies.

(59) The same few characters repeatedly appear in many of her early novels.

(60) Most of the boys and most of the girls in this party are such that each boy has chased at least half the girls and each girl has been chased by at least half the boys.

The adaptation of generalization 1 to 2-place quantifiers, needed in order to give these sentences precise interpretations, is analogous to (56).

We can verify the correctness of our interpretations by checking whether (57) to (60) can be put in the following canonical forms:

(61) Most of my right-hand gloves and most of my left-hand gloves

are such that each of the former matches exactly one of the latter and vice versa.

(62) Most of my friends and few of Truffaut's movies are such that each of the former saw at least two of the latter and each of the latter was seen by at least one of the former.

(63) Few characters and many of her early novels are such that each of the former appears in more than__ of the latter and each of the latter includes at least one of the former.

Sentence (60) is already in canonical form.

By replacing $\mathcal{Q}_1$ in generalization 1 with condition E below we get the *independent quantification* of section 6.

Condition E: each–some/some–each The pair $\langle X, Y \rangle$ is a maximal pair such that each element of X stands in the relation $\Phi^{\mathfrak{A}}$ to some element of Y and for each element of Y there is some element of X that stands to it in the relation $\Phi^{\mathfrak{A}}$.

Thus, both independent branching quantifiers and complex, Henkin-Barwise branching quantifiers fall under the general schema.

The second generalization abstracts from the outermost existential condition:

GENERALIZATION 2 A branching formula of the form (47) is true in a model $\mathfrak{A}$ with universe A iff there are $\mathcal{Q}_2$ pairs $\langle X, Y \rangle$ of subsets of the universe satisfying the maximal quantifier condition $\mathcal{Q}_1$ with respect to $\Phi^{\mathfrak{A}}$ such that X satisfies Q_1 and Y satisfies Q_2.[30]

The following sentences exemplify generalization 2 by substituting "by and large" (interpreted as "most") and "at most few" for $\mathcal{Q}_2$ ($\mathcal{Q}_1$ is the *each–all* condition):

(64) *By and large*, no more than a few boys and a few girls all date one another.

(65) There are *at most few* cases of more than a couple Eastern delegates and more than a couple Western delegates who are all on speaking terms with one another.[31]

The family of branching structures delineated above enlarges considerably the array of interpretations available for natural-language sentences with multiple quantifiers. The task of selecting the right alternative for a given natural-language quantification is easier if explicit inner and outer quantifier conditions occur in the sentence, but is more complicated other-

wise. One could, of course, be assisted by "context," but linguists will be interested in formulating general guidelines that hold across contexts. Indeed, we may look at Barwise's claims regarding monotone-increasing and monotone-decreasing English branching quantifiers in this light. According to Barwise, in English monotone-increasing branching quantifiers are usually accompanied by an inner "all," indicating a complex "*each all* structure" (with "some" as the outer quantifier condition); monotone-decreasing quantifiers are usually not accompanied by an inner quantifier condition, pointing to an independent (*each–some/some–each*) structure. These conjectures can be expressed in terms of my general definition schema of branching quantification (generalization 2).[32] However, the new multiplicity of inner and outer quantifier conditions introduced in the present section calls for refinement and supplementation of Barwise's conjectures.

8 Conclusion

My investigation has yielded a general definition schema for a pair of branching, or partially ordered, generalized quantifiers. The existing definitions, due to Barwise, constitute particular instances of this schema. The next task is to extend the schema, or particular instances thereof, especially (49), to arbitrarily large partially ordered quantifier prefixes. This task, however, is beyond the scope of the present work.

In "Branching Quantifiers and Natural Language" (1987), D. Westerståhl proposed a general definition of (Barwise's) branching quantifiers different from the ones suggested here. Although Westerståhl's motivation was similar to mine (dissatisfaction with the multiplicity of partial definitions), he approached the problem in a different way. Accepting Barwise's definitions of monotone-increasing and monotone-decreasing branching quantifiers, along with van Benthem's definition of branching non-monotonic quantifiers of the form "exactly n," Westerståhl constructed a general formula that yields the above definitions when the quantifiers plugged in have the "right" kind of monotonicity. That is, Westerståhl was looking for an umbrella under which the various partial existent definitions would fall. From the point of view of the issues discussed here, Westerståhl's approach is very similar to Barwise's. For that reason I did not include a separate discussion of his approach.[33] As for van Benthem's proposal for the analysis of non-monotonic branching quantifiers, his definition is

$$(66)\quad \begin{array}{l}(\text{Exactly-}n\,x)\cdot Ax, \\ \qquad\qquad\cdot \\ (\text{Exactly-}m\,y)\cdot By,\end{array} \quad Rxy =_{\mathrm{df}} (\exists X)(\exists Y)(X \subseteq A \;\&\; Y \subseteq B \;\&\; |X| = n \;\&\; |Y| = m \;\&\; R = X \times Y).$$[34]

For 1-place quantifiers, the definition would be

$$(67)\quad \begin{array}{l}(\text{Exactly-}n\,x) \\ (\text{Exactly-}m\,y)\end{array}\!\!> Rxy =_{\mathrm{df}} (\exists X)(\exists Y)(|X| = n \;\&\; |Y| = m \;\&\; R = X \times Y).$$

Since (67) is equivalent to (68) when R is not empty, I can express van Benthem's proposal in terms of my second generalization by saying that quantifiers of the form "exactly n" tend to occur in complex quantifications in which $\mathcal{Q}_1$ is "each–all" and $\mathcal{Q}_2$ is "the (only)."

(68) *The* (*only*) pair $\langle X, Y\rangle$ of subsets of the universe satisfying the *maximal each-all* condition with respect to R is such that X has *exactly n* elements and Y has *exactly m* elements.

I would like to end with a few general notes. Russell, recall, divided the enterprise of logic into two parts: the discovery of universal "templates" of truth and the discovery of new, philosophically significant logical forms. Branching quantifiers offer a striking example of an altogether new logico-linguistic form unlike anything thought to belong to language before Henkin's paper. One cannot, however, avoid asking: When does a generalization of a particular linguistic structure lead to a new, more general form of *language* and when does it end in a formal system that can no longer be considered language? Henkin, for instance, mentioned the possibility of constructing a densely ordered quantifier prefix. Would this be considered language? What about a prefix of quantifiers organized in some non-ordering pattern? Even the thoroughly studied form of an infinitely long linear prefix has yet to be evaluated with respect to our general concept of language.

Another question concerns the possibility of "importing" new structures into natural language. New forms continuously "appear" in all branches of mathematics and abstract logic. The "discovery" of branching prefixes in English makes one wonder whether new constructions cannot be introduced into natural language as well. Let us look back at Hintikka's "revelation" that branching quantifiers exist in English. Did Hintikka discover that all along we were talking about villagers' and townsmen's relatives hating each other *en masse* (each-all hatred) when we said that some relative of each villager and some relative of each townsman hate each other? Or did he, perhaps, propose to give a new meaning to a

syntactically well-formed but semantically empty (loosely defined) linguistic form? I am not sure what the right answer to this question is. Some of the English examples discussed in the literature strike me as having had a clear branching meaning even before the official seal of "branching quantification" was affixed to them. But others impress me as having been hopelessly vague before the advent of branching theory. These could have been semantically undetermined structures, forms in quest of content. Present-day languages have not used up all their lexical resources. Is logical form another unexhausted resource?

Investigations of the branching structure in the context of "generalized" logic led Barwise to extend Henkin's theory. My own inquiries have led to an even broader approach. In the next chapter I will return to the general conception of logic developed in this book and introduce some of its philosophical consequences. The philosophical ramifications of "unrestricted" logic have never before been (publicly) investigated. I will briefly point the direction of some philosophical inquiries and spell out a few results.

Chapter 6
A New Conception of Logic

The broad questions underlying this work concern the scope and limits of logic. Are the principles underlying modern logic fully exhausted by the standard system? Do generalized quantifiers signify a genuine breakthrough in logic? What are the boundaries of logic from the point of view of modern semantics? Starting with a general outlook of logic, I proceeded to examine Mostowski's generalization of the standard quantifiers, tracing its origins to Frege's interpretation of number statements. I then used Mostowski's theory as a jumping board for investigating the notion of "logicality." The initially loose philosophical question regarding the principles of logic received specific content: What makes a linguistic expression into a logical term? What are all the logical terms? My method of answering this question was conceptual. Examining Tarski's foundational work in semantics, I was able to identify a central motivation for constructing logic as a syntactic-semantic system in which logical truths and consequences are determined by reference to a full-blown system of models. I showed that within the framework of model-theoretic semantics the success of the logical project depends on the choice of logical terms. Inasmuch as logical constants represent the formal and necessary constituents of possible states of affairs, the system will accomplish its task. But the task is fully accomplished only if all formal and necessary constituents are taken into account. The standard system carries us one step toward the goal. It takes the full range of Tarskian or first-order Unrestricted Logic (UL) to achieve the objective in full. This outlook on logic is realized by logicians working within the dynamic field called "abstract" logic. It is also reflected in the work of linguists seeking to enhance the resources for studying the logical structure of natural language.

If the central claim of this book is correct, namely that standard mathematical logic, with its limited set of logical constants, does not fully express

the idea of logic, the question arises of whether a conceptual revision in the "official" doctrine is called for. Should "unrestricted logic" become "standard" logic? Because of the prominent place of standard first-order logic not only in mathematics but also in philosophy, linguistics, and related disciplines, at stake is a change in a very general and basic conceptual scheme. What are the philosophical ramifications of the new conception of logic? What new light does it shed on old philosophical questions? Are the conditions ripe for an "official" revision? And how should the new developments in semantics be viewed from the standpoint of proof theory? I would like to end this work with reflections on some aspects of these questions.

1 Revision in Logic

Putnam has convincingly argued that a change in a deeply ingrained conceptual scheme is seriously entertainable only if a well-developed alternative already exists. Referring to the revolution in geometry, Putnam argued that the laws of Euclidean geometry could not have been abandoned "before someone had worked out non-Euclidean geometry. That is to say, it is inconceivable that a scientist living in the time of Hume might have come to the conclusion that the laws of Euclidean geometry are false: 'I do not know what geometrical laws are true but I know the laws of Euclidean geometry are false.'"[1] Principles at the very center of our conceptual system are not overthrown unless "a rival theory is available."[2]

Is there a serious alternative to standard logical theory incorporating the principles of Unrestricted Logic delineated in this book? The unequivocal answer is yes. There exists a rich body of literature, in mathematics as well as in linguistics, in which nonstandard systems of first-order logic satisfying (UL) have been developed, studied, and applied. Mostowski's and Lindström's pioneering work led to a surge of logico-mathematical research. From Lindström's famous characterizations of "elementary logic" (1969) to works like Keisler's proof of the completeness of first-order logic with the quantifier "there exist uncountably many," the yield of mathematical investigations is astounding. For a representative collection of articles plus a comprehensive bibliography of more than a thousand items, the reader is referred to the 1985 volume *Model-Theoretic Logics*, edited by Barwise and Feferman.

In linguistics, Barwise and Cooper's 1981 paper also led to a profusion of literature. Generalized quantifiers became an essential component of formal semantics and of the theory of Logical Form within generative grammar. Representative works by van Benthem, Keenan, May, Westerståhl, and others are listed in the references.

Philosophically, the view that Tarskian or unrestricted logic is logic proper has not yet received its due attention. In particular, few philosophical arguments in support of the new view appeared in print. I hope this book has, to some extent, filled the void. If my argument is cogent, it will add a new philosophical dimension to the support that the new logic has received from other quarters.

2 The Logicist Thesis

The logicist thesis says that mathematics is reducible to logic in the sense that all mathematical theories can be formulated by purely logical means. That is, all mathematical constants are definable in terms of logical constants and all the theorems of (classical) mathematics are derivable from purely logical axioms by means of logical rules of derivation (and definitions). Now for the logicist thesis to be meaningful, the notions of logical constant, logical axiom, logical rule of derivation, and definition must be well defined and, moreover, so defined as to make the reduction nontrivial. In particular, it is essential that the reduction of mathematics to logic be carried out relative to a system of logic in which mathematical constants do not, in general, appear as primitive logical terms. The "fathers" of logicism did not engage in a critical examination of the concept of logical constant from this point of view. That is, they took it for granted that there is a small group of constants in terms of which the reduction is to be carried out: the truth-functional connectives, the existential (universal) quantifier, identity, and possibly the set-membership relation. The new conception of logic, however, contests this assumption. If my analysis of the semantic principles underlying modern logic in chapter 3 is correct, then any mathematical predicate or functor satisfying condition (E) can play the role of a primitive logical constant. Since mathematical constants in general satisfy (E) when defined as higher-level, the program of reducing mathematics to logic becomes trivial. Indeed, even if the whole of mathematics could be formulated within pure standard first-order logic, then (since the standard logical constants are nothing more than certain particular mathematical predicates) all that would have been accomplished is a reduction of some mathematical notions to others.

While the logicist program is meaningless from the point of view of the new conception of logic, its main tenet, that mathematical constants are essentially logical, is, of course, strongly supported by this conception.

Indeed, Russell's account of the logicality of mathematics in *Introduction to Mathematical Philosophy* is in complete agreement with my analysis:

There are words that express form. . . . And in every symbolization hitherto invented of mathematical logic there are symbols having constant formal meanings. . . . Such words or symbols express what are called 'logical constants.' Logical constants may be defined exactly as we defined forms; in fact, they are in essence the same thing. . . . In this sense all the 'constants' that occur in pure mathematics are logical constants.[3]

The difference between the new conception and the "old" logicism regarding mathematical constants is a matter of perspective. Both approaches are based on the equation that being mathematical = being formal = being logical. But while the classical logicists say that mathematical constants are essentially logical, the new conception implies that logical constants are esentially mathematical. Thus if the classical thesis is "the logicist thesis of mathematics," the new one is "the mathematical thesis of logic." Another point of difference worth noting is that according to the new conception, mathematical constants are logical only when construed as higher-level. Accordingly, the natural numbers, as individuals, are not logical objects. But as second-level entities, classes of classes, they are. This view is, as we saw in chapter 2, in some respects very Fregean. Frege's *logical* definition of the natural numbers takes numbers to be higher-level entities, i.e., classes of classes or classes of concepts. Indeed, the formulation of numerical statements as first-order quantifications in UL is exactly the same as Frege's in *The Foundations of Arithmetic*.

3 Mathematics and Logic

My discussion of logicism above highlighted one aspect of the relationship between logic and mathematics: in the new conception of logic any mathematical constant can play the role of a logical term, subject to certain requirements on its syntactic and semantic definitions. However, mathematical constants appear in the new logic also as extralogical constants, and this reflects another side of the relationship between logic and mathematics: as logical terms, mathematical constants are constituents of logical frameworks in which theories of various kinds are formulated and their logical consequences are drawn. But the "pool" of formal terms that can figure as logical constants is created in mathematics. The semantic definition of, say, the logical quantifier "there are uncountably many x" is based on some mathematical theory of sets. Similarly, the semantic definition of the quantifier "there is an odd number of x" is based on arithmetic. And even in standard logic, the semantic definitions of the truth-functional connectives and the universal (existential) quantifier are based on certain simple Boolean algebras. These observations point to a difference between logic and mathematics vis-à-vis formal terms: formal terms are *created* in mathematics; they are *used* in logic.

Now since logic provides a framework for theories in general, the meaning of formal terms can be given by a mathematical theory formulated within logic. We can thus picture the interplay between logic and mathematics as a cumulative process of definition and application. Starting with a logical system that applies certain elementary but powerful mathematical functions (Boolean truth functions, the universal/existential-quantifier function and, usually, identity) to a first-level extralogical vocabulary, we construct various formal theories. Such theories describe mathematical structures by delimiting the semantic variability of the extralogical terms of the language. This is done by introducing a set of extralogical axioms that partition the "universe" of all models for the language into those that do, and those that do not, "realize" the theory. In this way the axioms of the theory give specific meanings to all nonlogical terms of the language. Once mathematical terms are defined within the framework of standard first-order logic, they can be incorporated in the superstructure of a new, extended system of logic. As an example, consider the first-order theory of Peano arithmetic. As soon as arithmetic terms receive their meaning within this theory, we can convert them into logical arithmetic quantifiers: the numerical quantifiers, the "even" quantifier, quantitative comparative quantifiers ("there are fewer x's such that ... than x's such that ____"), and so on. We can now use the new logical vocabulary to formulate theories—mathematical, physical, etc.—that assume the existence of a machinery for counting and comparing sizes. In these theories we will *logically* conclude that, say, there are 4 *B*s, given that there are 2 *C*s, and that the number of *B*s is twice the number of *C*s. As we shall see below, there is an essential difference between applying mathematics by using mathematical terms as part of the logical superstructure and applying mathematics by adding extralogical mathematical constants and axioms to a theory of standard first-order logic.

4 Ontological Commitments of Theories

Quine is known for the thesis that the logical structure of theories in a standard first-order formalization reflects their ontological commitments. To determine the ontology of a theory $\mathcal{T}$ formulated in natural language (or a scientific "dialect" thereof), we formalize it as a (standard) first-order theory, $\mathcal{T}_1$, and examine those models of $\mathcal{T}_1$ in which the extralogical terms receive their intended meaning. $\mathcal{T}$ is committed to the existence of such objects as populate the universes of the intended model(s) of $\mathcal{T}_1$. Thus if $\mathcal{T}$ includes a sentence of the form

(1) Uncountably many things have the property P,

then, since the notion of uncountably many is not definable in pure standard first-order logic, we have to include in $\mathcal{T}_1$ some theory in which "uncountably many" can be defined. Choosing a set theory with *Ur*-elements, we express (1) as

(2) $(\exists x)[x$ is a set & x is uncountable &
$(\forall y)(y \in x \rightarrow y$ is an individual & $Py)]$.

And through (2), $\mathcal{T}$ is committed to the existence of sets.

Now, consider what happens if we formalize $\mathcal{T}$ within the framework of UL, using a system $\mathcal{L}$ that contains, in addition to the standard logical terms and axioms, the logical quantifier "uncountably many" and appropriate axioms (e.g., Keisler's). Obviously, we do not need set theory to express (1) in $\mathcal{L}$. The meaning of (1) is adequately captured by the sentence

(3) (Uncountably many $x)Px$,

which does not commit $\mathcal{T}$ to the existence of sets. So with a "right" choice of logical vocabulary, $\mathcal{T}$ can be formalized by a theory, $\mathcal{T}_2$, whose ontology consists merely of individuals, not sets.

We see that the new conception of logic allows us to save on ontology by augmenting the logical machinery. We can weaken the ontological commitments of theories by parsing more terms as logical. We no longer talk about *the* ontological commitment of an unformalized (or preformalized) theory $\mathcal{T}$ (there is no such thing!). Instead, ontological considerations become a factor in choosing logical frameworks for formalizing theories.

The examination of Quine's principle from the perspective of UL reveals the relativistic nature of his criterion. The comparison of $\mathcal{T}_1$ and $\mathcal{T}_2$ highlights the crucial role played by logical constants in deciding commitment in other theories of logic and ontology as well. Consider the simple, straightforward view that the commitment of a theory under a formalization $\mathcal{F}$ is determined by what is common to all models of $\mathcal{F}$. Here too the difference in logical terms between the formalizations $\mathcal{T}_1$ and $\mathcal{T}_2$ of $\mathcal{T}$ results in essentially different commitments. The occurrence of the quantifier "uncountably many" in (3) ensures that in every model of

$\mathscr{T}_2$, P is assigned an uncountable set of individuals. But by the Löwenheim-Skolem theorem, $\mathscr{T}_1$ has at least one model in which the predicate "x is uncountable" is given a nonstandard interpretation and P is assigned a countable set. We can thus say that $\mathscr{T}_2$ is committed to an ontology of uncountably many objects, whereas $\mathscr{T}_1$ is not.

We see that logical terms are vehicles of *strong ontological commitment*, while extralogical terms transmit a relatively weak commitment. This difference in ontological import between logical and extralogical terms is explained by the fact that logical terms are semantically pre-fixed, whereas the meaning of extralogical terms is relative to models. To use Putnam's turn of speech, extralogical terms are viewed from *within* models, whereas logical terms are viewed from the *outside*.[4] Including formal terms as part of the logical superstructure allows us to use them in the logic with "a view from the outside."

The distinction between "strong" and "weak" ontological commitments explains the difference between using mathematical notions as part of the logical machinery and using them as extralogical terms in theories within the logic. It also suggests a guideline for choosing logical frameworks. If you formulate, say, a physical theory and you want to use formal tools created elsewhere (i.e., in some mathematical theory), you might as well include the mathematical apparatus as part of the logical superstructure. This will reflect the fact that you are not interested in specifying the meanings of the mathematical terms but in saying something about the physical world, using mathematical notions which you take as given. The pre-fixed notions will enable you to make some very strong claims about the physical world, strong in the sense that what they say does not vary from one model of the theory to another. All this will be done without compromising the usefulness of the logical framework in determining necessary and formal consequences. If, on the other hand, your goal is to define the mathematical notions themselves, you cannot construe them as logical, because as such their meaning would have to be given at the outset. You have to use undefined terms of the language (i.e., extralogical terms) and then construct a theory that will give these notions a distinctive content.[5]

5 Metaphysics and Logic

What role, if any, does metaphysics play in logics based on Tarski's ideas? First, for Tarski, the very notion of semantics has a strong metaphysical

connotation. Semantics investigates concepts having to do with the relationship between language and the world (see page 39). The categories used in classifying relevant features of the world are, ipso facto, an important factor in the analysis of such concepts. More specifically, as we have seen earlier in the book, it is crucial for Tarski that an adequate system of logic yield consequences that hold *necessarily* of reality. In that way metaphysics provides an important criterion for evaluating logical systems vis-à-vis their goal. But the role of metaphysics does not end with this external criterion. To see the metaphysical dimension of Tarskian semantics more clearly, it might be well to contrast his model-theoretic method with another type of theory, which, following Etchemendy 1990, I will call "interpretational." The interesting feature of *interpretational semantics* from my point of view is that it purports to ensure the satisfaction of Tarski's metaphysical condition by purely *syntactic* means. The interpretational definition of "logical consequence" is the following:

DEFINITION LC″ The sentence X is a *logical consequence* of the set of sentences K iff there is no permissible substitution for the nonlogical terms in the sentences of K and in X that makes all the former true and the latter false.

(A substitution is permissible if it is uniform and it preserves syntactic categories.) This definition, in essence, goes back to Bolzano (1837). It can also be found in modern texts, e.g., Quine's *Philosophy of Logic* (1970).

The distinctive feature of the interpretational test for logical consequence is that it is based on substitution of strings of symbols. Definition (LC″) does not take into account anything but grammar and the distribution of truth values to all the sentences of the language. Thus to the extent that syntactic analysis and a list of truth values are all that are needed to determine logical truths and consequences, interpretational semantics has nothing to do with metaphysics.

Tarski rejected the substitutional definition of "logical consequence" just for that reason. The success of interpretational semantics depends on the expressive power of the language. Relevant possible states of affairs may not be taken into account if the language is too poor to describe them. Thus, consider a language in which the only primitive nonlogical terms are the individual constants "Sartre" and "Camus" and the predicates "x is active in the French Resistance" and "x is a novelist." In this language the sentence

(4) Sartre was active in the French Resistance

will come out logically true under the substitutional test. But obviously, (4) is not necessarily true.

Etchemendy pointed out another problem with interpretational theory due to its syntactic character.[6] In interpretational "semantics," as in model-theoretic semantics, "logical consequence" and the other logical concepts are defined relative to a set of logical constants. But in interpretational semantics, the set of logical constants is an *arbitrary* set of terms, arbitrary because the interpretational theory does not offer a guide for determining whether a term is logical or not. Logical and extralogical terms are defined by use, and for all that interpretational semantics has to say, any term might be used either way. What Quine calls the remarkable concurrence of the substitutional and model-theoretic definitions of "logical consequence" for standard first-order logic is no more than a "happy" accident.[7] Since the standard logical constants do not form a grammatically distinct group, they are, from the point of view of interpretational semantics, indistinguishable from other terms that can also be held constant in the substitutional test. Thus even if every individual, property, and relation "participating" in relevant possible states of affairs has a name in the language, some divisions of terms into the logical and extralogical will yield unacceptable results. Suppose, for instance, that expressions naming Sartre and the property of being active in the French Resistance are included in the set of *fixed* (i.e., logical) terms. Then (4) will again turn out to be logically true. (See chapter 3.)

Tarski's semantics avoids the two problems indicated above by using a semantic apparatus which allows us to represent the relationship between language and the world in a way that distinguishes formal and necessary features of reality. The main semantic tool is the *model*, whose role is to represent possible states of affairs relative to a given language. Since *any* set of objects together with an "interpretation" of the nonlogical terms within the set determine a model, every possible state of affairs vis-à-vis the extralogical vocabulary is represented (extensionally). Furthermore, the choice of logical constants is constrained by the requirement that the logical superstructure represent formal, metaphysically unchanging parameters of possible states of affairs. (It should be noted that "possibility" in this context is "formal possibility." Therefore, the totality of models reflects "possibilities" that in general metaphysics might be ruled out by nonformal considerations. That is to say, the notion of possibility underlying the choice of models is wider than in metaphysics proper.)

Although metaphysical considerations are central to Tarskian semantics, only the most basic and general metaphysical principles are taken into account. The historical Tarski expressed a dislike for "abstruse" philosophical theories. The notions of necessity and possibility he used were, he emphasized, the common, everyday notions, not the philosopher's. I think Tarski's mistrust of philosophy is not warranted, but the claim that the philosophical foundation of logic should not rest on the web of philosophical controversies regarding modalities appears to me sound. Thus the view underlying the new conception of logic, that the mathematical "coordinates" of reality do not change from one possible world to another (and therefore mathematical constants can, in general, play the role of logical constants), is based on a basic, generally accepted belief about the nature of reality.

We cannot rule out, however, divergence of opinions even with respect to "core" metaphysical principles. And for those who do not share the "common" belief regarding the nature of mathematical properties, I propose the following relativistic view of logic: we can look at the definition of "logical terms" in chapter 3 as a schema saying that to treat a term as logical is to take it as naming a rigid, formal property or function (fixed across possible states of affairs) and define it in accordance with conditions (C) to (E). It is then left for the user to determine whether or not it is appropriate to treat a given term in that way. (A similar strategy will enable one to reconcile nominalistic compunctions with the new conception: depending on the metalinguistic resources one finds acceptable, one will construe those mathematical predicates that are definable in one's language as logical constants.)

The foundations of Tarskian semantics reach deep into metaphysics, but the link between models and reality may have some weak joints. In particular, Tarski has never shown that the set-theoretic structures that make up models constitute adequate representations of all (formally) possible states of affairs. This issue is beyond the scope of the present book, but two questions that may arise are the following: Is it formally necessary that reality consist of discrete, countable objects of the kind that can be represented by *Ur*-elements (or other constituents) of a standard set theory? Does the standard model-theoretic description of all possible states of affairs have enough parameters to represent all relevant aspects of possible situations (relevant, that is, for the identification of formally necessary consequences)? These and similar questions lie at the bottom of nonstandard models for physics, probablistic logic, and, if we put aside formality, such discourse theories as "situation semantics."

6 Proof-Theoretic Perspective

The philosophical justification of the new conception of logic is based on an analysis of certain *semantic* principles underlying modern logic. What about proof theory? Should we not set proof-theoretic standards for an adequate system of logic, for example, that it be *complete* relative to an "acceptable" deductive apparatus? The new logic, one would then object, surely fails to comply with this requirement! I think this judgement is premature. The "new conception of logic" is a result of reexamining the philosophical ideas behind logical semantics in response to certain mathematical generalizations of standard semantic notions (Mostowski and others). There is no sense in comparing the generalized semantics with current un- or pre-generalized proof theory. To do justice to the new conception from a proof-theoretic perspective, one has to cast a new, critical look at the standard notion of proof. This task may be exacting because there is no body of mathematical generalizations in proof theory directly parallel to "generalized logic" in contemporary model theory. However, if the new philosophical extension of logic based on semantics is significant, it poses a challenge to proof theory that cannot be overlooked. I can put it this way: if Tarski is right about the basic intuitions underlying our conception of logical truth and consequence, and if my analysis is correct, namely that these intuitions are not exhausted by standard first-order semantics, then since standard first-order logic has equal semantic and proof-theoretic power (completeness), these intuitions are not exhausted by standard first-order proof theory either. Semantically, we have seen, it suffices to enrich the superstructure of first-order logic by adding new logical terms. But what has to be done proof-theoretically? I hope that future researchers will take up this question as a challenge.

Appendix

Chapter 2,[1] Section 2

DEFINITION 1 Let A be a set. A *quantifier on A* is a function

$q : P(A) \rightarrow \{T, F\}$

such that if $m : A \rightarrow A$ is an automorphism (permutation) of A, i.e., m is one-to-one and onto A, then for every $B \subseteq A$,

$q(m(B)) = q(B)$,

where $m(B)$ is the image of B under m.

It is easy to see that Boolean combinations of quantifiers on A are also quantifiers on A.

DEFINITION 2 Let α be a cardinal number. A *2-partition of α* is a pair of cardinals $\langle \beta, \gamma \rangle$ such that $\beta + \gamma = \alpha$.

DEFINITION 3 Let $\langle \beta, \gamma \rangle_\alpha$ be the class of 2-partitions of α. A *cardinality function on 2-partitions of α* is a function $t : \langle \beta, \gamma \rangle_\alpha \rightarrow \{T, F\}$.

THEOREM 1 (Mostowski 1957.) Let A be a set. Let $\mathscr{T}$ be the set of cardinality functions on 2-partitions of $\alpha = |A|$. Let $\mathscr{Q}$ be the set of quantifiers on A. Then there exists a one-to-one function h from $\mathscr{T}$ onto $\mathscr{Q}$ defined as follows:

For any $t \in \mathscr{T}$, $h(t) =$ the quantifier q on A such that for any $B \subseteq A$,

$q(B) = t(|B|, |A - B|)$.

I will symbolize a quantifier q on A as Q_A. Given a quantifier on A, Q_A, I will call the cardinality function t satisfying the above equation the *cardinality counterpart* of Q_A and symbolize it as t^Q_A.

DEFINITION 4 A *quantifier* Q is a function that assigns to each universe A a quantifier on A, Q_A, and such that if A, A' are universes of the same cardinality, then Q_A and $Q_{A'}$ have the same cardinality counterpart.

Chapter 2, Section 5

DEFINITION 5 Let A be a set. A *2-place quantifier on A* is a function

$q : \boldsymbol{P}(A) \times \boldsymbol{P}(A) \rightarrow \{T, F\}$

such that if $m : A \rightarrow A$ is an automorphism of A, then for every $B, C \subseteq A$,

$q(m(B), m(C)) = q(B, C)$,

where $m(B)$ and $m(C)$ are the images of B and C under m.

DEFINITION 6 Let α be a cardinal number. A *4-partition of* α is a quadruple $(\beta, \gamma, \delta, \varepsilon)$ of cardinals such that $\beta + \gamma + \delta + \varepsilon = \alpha$.

DEFINITION 7 Let $(\beta, \gamma, \delta, \varepsilon)_\alpha$ be the class of 4-partitions of α. A *cardinality function on 4-partitions of* α is a function

$t : (\beta, \gamma, \delta, \varepsilon)_\alpha \rightarrow \{T, F\}$.

THEOREM 2 (Lindström 1966.) Let A be a set. Let $\mathscr{T}$ be the set of cardinality functions on 4-partitions of $\alpha = |A|$. Let $\mathscr{Q}$ be the set of 2-place quantifiers on A. Then there exists a one-to-one function h from $\mathscr{T}$ onto $\mathscr{Q}$ defined as follows:

For any $t \in \mathscr{T}$, $h(t) =$ the 2-place quantifier q on A such that for any $B, C \subseteq A$, $q(B, C) = t(|B \cap C|, |B - C|, |C - B|, |A - (B \cup C)|)$.

Given a 2-place quantifier on A, Q^2, I will call the cardinality function t satisfying the above equation the *cardinality counterpart of* Q^2 and symbolize it as t^Q.

DEFINITION 8 A *2-place quantifier* Q^2 is a function that assigns to each universe A a 2-place quantifier on A, Q^2_A, and such that if A, A' are universes of the same cardinality, then Q^2_A and $Q^2_{A'}$ have the same cardinality counterpart.

Chapter 4, Section 2

Proof of theorem 1 The proof is straightforward because we have already introduced all the concepts connecting the ordinal structures over which α-operators are defined with structures within $\mathfrak{A}$ over which logical terms restricted to $\mathfrak{A}$ are defined. I will prove

1. h is a funtion from $\mathcal{O}_\alpha$ into $\mathscr{C}|\mathfrak{A}$,
2. h is onto $\mathscr{C}|\mathfrak{A}$, and
3. h is one-to-one.

(1.a) First I prove that h is a function. Let o_α be an α-operator of type $\langle t_1, \ldots, t_k\rangle$. Let $\langle s_1, \ldots, s_k\rangle$ be a sequence such that for $1 \leq i \leq k$, $s_i \in A$ if $t_i = 0$, and $s_i \subseteq A^n$ if $t_i = n \neq 0$. I have to show that $[\langle i(s_1), \ldots, i(s_k)\rangle]$ exists and is unique. Existence is obvious. To prove uniqueness, let I, I' be two indexings of A by α. Let $i(s_1), \ldots, i(s_k)$ and $i'(s_1), \ldots, i'(s_k)$ be the index images of $s_1, \ldots, s_k$ under I and I' respectively. $I'^{-1} \circ I$ is a permutation of α and $\langle i'(s_1), \ldots, i'(s_k)\rangle$ is the image of $\langle i(s_1), \ldots, i(s_k)\rangle$ under $I'^{-1} \circ I$. Hence, $\langle i(s_1), \ldots, i(s_k)\rangle$ and $\langle i'(s_1), \ldots, i'(s_k)\rangle$ are similar and $[\langle i(s_1), \ldots, i(s_k)\rangle] = [\langle i'(s_1), \ldots, i'(s_k)\rangle]$. That is, $[\langle i(s_1), \ldots, i(s_k)\rangle]$ is unique.

(1.b) Next I prove that h is into $\mathscr{C}|\mathfrak{A}$. Let o_α be an α-operator of type $\langle t_1, \ldots, t_k\rangle$. Let $B_1 \times \cdots \times B_k$ be a Cartesian product such that for $1 \leq i \leq k$, $B_i = A$ if $t_i = 0$, and $B_i = \boldsymbol{P}(A^n)$ if $t_i = n \neq 0$. Let $C_\mathfrak{A}$ be a function from $B_1 \times \cdots \times B_k$ into $\{T, F\}$ such that for every $\langle s_1, \ldots, s_k\rangle \in \mathrm{Dom}(C_\mathfrak{A})$, $C_\mathfrak{A}(s_1, \ldots, s_k) = o_\alpha[\langle i(s_1), \ldots, i(s_k)\rangle]$, where for some indexing I of A by α, $i(s_j)$, $1 \leq j \leq k$, is the index image of s_j under I. By definition of h, $C_\mathfrak{A} = h(o_\alpha)$. (By (1.a) above, $C_\mathfrak{A}$ is well defined.) We have to show that $C_\mathfrak{A}$ is indeed a logical term restricted to $\mathfrak{A}$. In particular, we have to show that $C_\mathfrak{A}$ satisfies the restriction of condition (E) of chapter 3, section 6 to $\mathfrak{A}$. That is, if $\langle s_1, \ldots, s_k\rangle$ and $\langle s'_1, \ldots, s'_k\rangle$ are in $\mathrm{Dom}(C_\mathfrak{A})$ and $\langle A, s_1, \ldots, s_k\rangle \cong \langle A, s'_1, \ldots, s'_k\rangle$, then $C_\mathfrak{A}\langle s_1, \ldots, s_k\rangle = C_\mathfrak{A}(s'_1, \ldots, s'_k)$. Take any indexing I of A by α. For $1 \leq j \leq k$, let $i(s_j)$ be the index image of s_j under I. By definition,

$C_\mathfrak{A}(s_1, \ldots, s_k) = o_\alpha[\langle i(s_1), \ldots, i(s_k)\rangle]$,

$C_\mathfrak{A}(s'_1, \ldots, s'_k) = o_\alpha[\langle i(s'_1), \ldots, i(s'_k)\rangle]$.

It suffices to show that $[\langle i(s_1), \ldots, i(s_k)\rangle] = [\langle i(s'_1), \ldots, i(s'_k)\rangle]$. Let f be an isomorphism of $\langle A, s_1, \ldots, s_k\rangle$ onto $\langle A, s'_1, \ldots, s'_k\rangle$. Thus f is a permutation of A. Define a permutation m of α as follows: for all $\beta \in \alpha$,

$m(\beta) = \gamma$ iff $f(a_\beta) = a_\gamma$.

Clearly, $\langle i(s'_1), \ldots, i(s'_k)\rangle$ is the image of $\langle i(s_1), \ldots, i(s_k)\rangle$ under the permutation induced by m. Hence, $\langle i(s_1), \ldots, i(s_k)\rangle$ and $\langle i(s'_1), \ldots, i(s'_k)\rangle$ are similar. Therefore, $[\langle i(s_1), \ldots, i(s_k)\rangle] = [\langle i(s'_1), \ldots, i(s'_k)\rangle]$.

(2) The next step is to prove that h is onto $\mathscr{C}|\mathfrak{A}$. Take any $C_\mathfrak{A} \in \mathscr{C}|\mathfrak{A}$. The claim is that there is an $o_\alpha \in \mathcal{O}_\alpha$ such that $h(o_\alpha) = C_\mathfrak{A}$. Let the type of

$C_{\mathfrak{A}}$ be $\langle t_1, \ldots, t_k \rangle$. Consider the totality of α arguments $R(\alpha) = \langle r_1(\alpha), \ldots, r_k(\alpha)\rangle$ of the same mark. Let $[\mathfrak{R}(\alpha)]$ be the set of all equivalence classes of $R(\alpha)$'s under the relation of similarity. Define

$o_\alpha : [\mathfrak{R}(\alpha)] \to \{\mathrm{T}, \mathrm{F}\}$

as follows: Let I be some indexing of A by α. For any $[\langle r_1(\alpha), \ldots, r_k(\alpha)\rangle] \in \mathrm{Dom}(o_\alpha)$, $o_\alpha[\langle r_1(\alpha), \ldots, r_k(\alpha)\rangle] = \mathrm{T}$ iff for some structure $\langle A, s_1, \ldots, s_k\rangle$ such that s_i, $1 \leq i \leq k$, is of mark t_i, (1) and (2) below hold:

(1) $r_j(\alpha) = i(s_j)$, where $1 \leq j \leq k$, and $i(s_j)$ is the index image of s_j under I

(2) $C_{\mathfrak{A}}(s_1, \ldots, s_k) = \mathrm{T}$

I have to show that o_α is a well-defined α-operator. Let $\langle r_1(\alpha), \ldots, r_k(\alpha)\rangle$ and $\langle r'_1(\alpha), \ldots, r'_k(\alpha)\rangle$ be two similar α-arguments of mark $\langle m_1, \ldots, m_k\rangle$. Let the corresponding structures within $\mathfrak{A}$ be $\langle s_1, \ldots, s_k\rangle$ and $\langle s'_1, \ldots, s'_k\rangle$ respectively. It suffices to show that $C_{\mathfrak{A}}(s_1, \ldots, s_k) = C_{\mathfrak{A}}(s'_1, \ldots, s'_k)$. But this follows from the fact that $C_{\mathfrak{A}}$ is a logical term (restricted to $\mathfrak{A}$) and the fact that the structures $\langle A, s_1, \ldots, s_k\rangle$ and $\langle A, s'_1, \ldots, s'_k\rangle$ are isomorphic. (That the structures $\langle A, s_1, \ldots, s_k\rangle$ and $\langle A, s'_1, \ldots, s'_k\rangle$ are isomorphic is obvious: By definition of similarity, there is a permutation m of α such that $\langle \alpha, r_1(\alpha), \ldots, r_k(\alpha)\rangle \cong \langle \alpha, r'_1(\alpha), \ldots, r'_k(\alpha)\rangle$, where for $1 \leq j \leq k$, $r'_j(\alpha)$ is the image of $r_j(\alpha)$ under m. Let f be the corresponding permutation of A. That is, under the given indexing $f(a_\beta) = a_\gamma$ iff $m(\beta) = \gamma$. Now $\langle A, s_1, \ldots, s_k\rangle \cong \langle \alpha, r_1(\alpha), \ldots, r_k(\alpha)\rangle$, and $\langle A, s'_1, \ldots, s'_k\rangle \cong \langle \alpha, r'_1(\alpha), \ldots, r'_k(\alpha)\rangle$. Therefore, $\langle A, s_1, \ldots, s_k\rangle \cong \langle A, s'_1, \ldots, s'_k\rangle$.)

(3) Finally, I prove that h is one-to-one. Let $o1_\alpha \neq o2_\alpha$. Then for some $R(\alpha)$, $o1_\alpha[R(\alpha)] \neq o2_\alpha[R(\alpha)]$. Say $o1_\alpha[R(\alpha)] = \mathrm{T}$, $o2_\alpha[R(\alpha)] = \mathrm{F}$. Suppose $h(o1_\alpha) = C1_{\mathfrak{A}}$, $h(o2_\alpha) = C2_{\mathfrak{A}}$. We have to show that $C1_{\mathfrak{A}} \neq C2_{\mathfrak{A}}$. Let I be an indexing of A by α and let s be the structure within $\mathfrak{A}$ determined by $R(\alpha)$ through I. Then by definition of h, $C1_{\mathfrak{A}}(s) = \mathrm{T}$, $C2_{\mathfrak{A}}(s) = \mathrm{F}$. Hence, $C1_{\mathfrak{A}} \neq C2_{\mathfrak{A}}$. Q.E.D.

Chapter 4, Section 4

Binary relational quantifiers satisfying the invariance condition (b.1)

DEFINITION 11 Let α be a cardinal number, identified with the least ordinal of cardinality α and defined as the set of all smaller ordinals (as in section 2). Let $(\beta, \gamma)_\alpha$ be the set of 2-cardinal-partitions of α, i.e., the set of all pairs of cardinals (β, γ) such that $\beta + \gamma = \alpha$ (+ being cardinal addition). Consider the functions

$f : \alpha \to (\beta, \gamma)_\alpha$,

and let $\mathscr{F}$ be the set of such functions. For any $f, f' \in \mathscr{F}$, the functions f and f' are *similar* iff there is an automorphism m of α such that for every $\delta \in \alpha$,

$f(\delta) = f'(m(\delta))$.

DEFINITION 12 Let α be a cardinal number. Consider the functions f defined above. Let $[\mathscr{F}]$ be the set of equivalence classes $[f]$ under the relation of similarity defined above. Then a *binary cardinality function on* α is a function

$k_\alpha : [\mathscr{F}] \to \{\mathrm{T}, \mathrm{F}\}$.

THEOREM 2 (Higginbotham and May 1981.) Let A be a set. Let $\mathscr{K}$ be the set of binary cardinality functions k_α on $\alpha = |A|$. Let $\mathscr{Q}$ be the set of 1-place quantifiers on binary relations over A satisfying the invariance condition (b.1) (p. 88). Then there exists a one-to-one function h from $\mathscr{K}$ onto $\mathscr{Q}$ defined thus:

For every $k_\alpha \in \mathscr{K}$, $h(k_\alpha) =$ the quantifier $q_A \in \mathscr{Q}$ such that for any $R \subseteq A^2$, $q_A(R) = k_\alpha([f_R])$, where $f_R : \alpha \to (\beta, \gamma)_\alpha$ is defined, relative to some one-to-one and onto indexing i of A by α, as follows: for every $\delta \in \alpha$, $f_R(\delta) = (\varepsilon, \zeta)$ iff $\varepsilon = |\{a : \langle a_\delta, a\rangle \in R\}|$ and $\zeta = |\{a : \langle a_\delta, a\rangle \notin R\}|$.

I say that f_R *represents the cardinalities of R in A* (relative to i).[2]

LEMMA 1 Let A be a set. Let m be a 1-automorphism of A^2, and let m_1 be an automorphism of A such that for all $a, b \in A$, $m(a, b) = (m_1(a), b')$ for some $b' \in A$. Let R, R' be two binary relations on A such that $R' = m(R)$. Then for every $a \in A$,

(1) $|\{b \in A : \langle a, b\rangle \in R\}| = |\{b' \in A : \langle m_1(a), b'\rangle \in R'\}|$,

(2) $|\{b \in A : \langle a, b\rangle \notin R\}| = |\{b' \in A : \langle m_1(a), b'\rangle \notin R'\}|$.

Proof Take any $a \in A$.

(1) Let $\{b \in A : \langle a, b\rangle \in R\} = B$, $\{b' \in A : \langle m_1(a), b'\rangle \in R'\} = B'$. I want to prove that $|B| = |B'|$. Let $C = \{c \in A : \text{for some } b \in B,\ m(a, b) = (m_1(a), c)\}$. Then $|B| = |C|$ because m is a 1-automorphism based on m_1. The claim is that $B' = C$. This follows from the fact that $R' = m(R)$. As a result, $|B| = |B'|$.

(2) The proof is similar to (1). Q.E.D.

Proof of theorem 2 Let the members of A be ordered with indices in α. (The index map $i : \alpha \to A$ is one-to-one and onto.) I will prove that

1. h is a well-defined function from $\mathscr{K}$ into $\mathscr{Q}$,
2. h is onto $\mathscr{Q}$,
3. h is one-to-one.

(1.a) First I show that h is a well-defined function. Let k_α be any binary cardinality function on α.

(1.a.i) $h(k_\alpha)$ exists. Let R be any binary relation included in A^2. Then R is represented by some function $f_R : \alpha \to (\beta, \gamma)_\alpha$. Since $k_\alpha([f_R])$ exists for every R, so does $h(k_\alpha)$.

(1.a.ii) $h(k_\alpha)$ is unique. Let R be any binary relation as above. Then R is *uniquely* represented by f_R under the given indexing. So $(h(k_\alpha))(R)$ is unique under the given indexing. Let f_R, f'_R be the representations of R under two indexings of A by α. Then clearly f_R and f'_R are similar. That is, $[f_R] = [f'_R]$. Hence, $(h(k_\alpha))(R)$ is unique.

(1.b) Next I show that h is into $\mathscr{Q}$. Let k_α, R, and f_R be as above. Let R' be any binary relation on A such that for some 1-automorphism m of A^2, $R' = m(R)$. Let $f_{R'}$ be the function representing the cardinalities of R' in A. I will show that $(h(k_\alpha))(R) = (h(k_\alpha))(R')$. It suffices to show that f_R and $f_{R'}$ are similar. Since m is a 1-automorphism, there is an automorphism m_1 of A such that for all $\delta, \eta \in \alpha$, $m(a_\delta, a_\eta) = (m_1(a_\delta), a_{\eta'})$ for some $\eta' \in \alpha$. Let ρ be an automorphism of α that simulates m_1. I.e., for all $\delta \in \alpha$, $\rho(\delta) =$ the $\theta \in \alpha$ such that $m_1(a_\delta) = a_\theta$. To show that f_R and $f_{R'}$ are similar, it suffices to show that for every $a_\delta \in A$, the number of elements to which a_δ stands in the relation R is the same as the number of elements to which $m_1(a_\delta)$ stands in the relation R', and similarly, the number of elements to which a_δ does not stand in the relation R is the same as the number of elements to which $m_1(a_\delta)$ does not stand in the relation R'. This is proved in lemma 1. As a result, $(h(k_\alpha))(R) = k_\alpha([f_R]) = k_\alpha([f_{R'}]) = (h(k_\alpha))(R')$.

(2) Let $q_A \in \mathscr{Q}$, and let $\mathfrak{R} = \{R \subseteq A^2 : q_A(R) = \mathrm{T}\}$. Define a binary cardinality function on α, $k1_\alpha$, as follows:

$k1_\alpha([f]) = \mathrm{T}$ iff for some $R \in \mathfrak{R}$, f represents R in A (under the given indexing).

Claim: $k1_\alpha$ is well defined. I have to show that if $f_R, f_{R'}$ are similar functions representing R and R' respectively, then $R \in \mathfrak{R}$ iff $R' \in \mathfrak{R}$. I will show that if $f_R, f_{R'}$ are similar, there is a 1-automorphism m of $A \times A$ such that $m(R) = R'$. Let ρ be an automorphism of α such that for every $\delta \in \alpha$, $f_R(\delta) = f_{R'}(\rho(\delta))$. Let m_1 be the automorphism of A induced by ρ (through the given indexing of A). I will define m as follows: Take any $a \in A$. Let $m_1(a) = a'$. Then since $f_R, f_{R'}$ are similar and $f_R, f_{R'}$ represent R, R' respectively,

$|B| = |B'|$ and $|\bar{B}| = |\bar{B}'|$,

where $B = \{b : \langle a, b\rangle \in R\}$, $B' = \{b' : \langle a', b'\rangle \in R'\}$, and $\bar{B}, \bar{B}'$ are the complements of B, B' (in A) respectively. Let m' be any one-to-one function from B onto B' and m'' any one-to-one function from $\bar{B}$ onto $\bar{B}'$. Let b be any member of A. I define the value of m for (a, b) as follows:

$$m(a, b) = \begin{cases} (a', m'(b)) & \text{if } b \in B \\ (a', m''(b)) & \text{otherwise} \end{cases}$$

(I assume the axiom of choice.) The reader can check that m is a well-defined 1-automorphism of $A \times A$. I have shown that $k1_\alpha$ is well defined. Clarly, $h(k1_\alpha) = q_A$.

(3) Suppose that for two distinct binary cardinality functions on α, $k1_\alpha$ and $k2_\alpha$, $h(k1_\alpha) = h(k2_\alpha) = q_A$. Since $k1_\alpha \neq k2_\alpha$, for some f, $k1_\alpha([f]) \neq k2_\alpha([f])$. Let R be a binary relation on A whose cardinalities are represented by f. (It is easy to describe the construction of R. For each $a_\delta \in A$, I construct the set $Ra_\delta = \{a : \langle a_\delta, a\rangle \in R\}$ as follows: Let $f(\delta) = (\varepsilon, \zeta)$. There are two cases: (a) $\varepsilon = \zeta$, (b) $\varepsilon < \zeta$ or $\zeta < \varepsilon$. Case (a). Construct Ra_δ by alternating $a_0 \in Ra_\delta$, $a_1 \notin Ra_\delta$, ..., $a_i \in Ra_\delta$, $a_{i+1} \notin Ra_\delta$, ... [If μ is a limit ordinal, let $a_\mu \in Ra_\delta$.] Case (b). Suppose $\varepsilon < \zeta$. Take the least ordinal ξ equipollent to ε. Define, for all $a_\iota \in A$, $a_\iota \in Ra_\delta$ iff $\iota < \xi$. If $\zeta < \varepsilon$, define the complement of Ra_δ in the same way.) Suppose (i) that $k1_\alpha([f]) = \mathrm{T}$ and (ii) that $k2_\alpha([f]) = \mathrm{F}$. Then by definition of q_A, (i) implies that $q_A(R) = \mathrm{T}$, while (ii) implies that $q_A(R) = \mathrm{F}$. But this is impossible, since q_A is a function. Q.E.D.

Notes

Chapter 1

1. *Our Knowledge of the External World*, p. 67.

2. *Our Knowledge of the External World*, p. 68.

3. See p. 12.

4. See Barwise and Feferman 1985 and the extensive bibliography there.

5. That makes it more suitable for the task in question than the simple (or ramified) theory of types, in which the logical constants are determined prior to the establishment of types.

6. Two first-order structures are *elementarily equivalent* iff they are indistinguishable by any sentence of standard first-order logic (elementary logic). That is, there is no first-order sentence true in one of the structures and false in the other.

7. P. 196.

8. A *Skolem function* is a function that represents an existential quantifier in a quantifier prefix of the form "$(\forall x_1) \ldots (\forall x_n)(\exists y)$," where $n \geq 1$. Thus a statement of the form "Every x stands to some y in the relation R" is logically equivalent to "There is a function f such that every x stands to $f(x)$ in the relation R."

9. Kant 1781/1787, p. 17.

10. See Hodes 1982, p. 162.

11. These are Hodes's reformulations of statements discussed by Frege in *The Foundations of Arithmetic* (1884), p. 69. See Hodes 1982, p. 170 and also Hodes 1984, p. 129.

12. Hodes 1984, p. 129.

13. Frege 1884, p. 69.

14. Carnap 1939, pp. 28–29.

15. Carnap 1950, p. 250.

16. Carnap 1950, p. 250.

17. For a partial selection, see bibliography.

Chapter 2

1. Mostowski 1957, p. 12.

2. Frege 1884, p. 65.

3. Frege 1884, p. 65.

4. Frege 1884, p. 64. Although Frege does not refer to quantifiers explicitly in these passages, it is clear from the relationship established in *Begriffsschrift* between the universal quantifier and statements of existence that "there exists" is to be understood as a quantifier.

5. Frege uses different styles of variables for quantification and for open sentences. The role of variables of quantification is to "express generality," while variables in open sentences "show the places where the completing sign has to be inserted" (Frege 1904, p. 114).

6. Frege 1884, p. 59.

7. See, e.g., Slomson 1976, pp. 249–250.

8. Mostowski 1957, p. 13.

9. Mostowski 1957, p. 13. The following constraints are to be taken for granted in the context of Mostowski's work: the second-level predicates in question are defined over all first-level 1-place predicates of the language, and they are extensional.

10. Dummett 1973, p. 22, n.

11. Barwise and Cooper 1981, p. 163. Sentences relabeled.

12. See Keisler 1977 and 1985.

13. For an extensive discussion of generalized quantifiers in the setting of nonstandard models and infinitistic languages, see Barwise and Feferman 1985.

14. See Mostowski 1957, p. 17.

15. Mostowski 1957, p. 12.

16. Barwise and Cooper 1981, p. 159.

17. Sentences (14) and (15) are (1.a) and (1.b) of Barwise and Cooper 1981, p. 160.

18. Here we have a formula with embedded generalized quantifiers. The definition of truth for such formulas is straightforward. Informally, if $\mathfrak{A}$ is a model with a finite universe A, then $(Mx)(My)\Phi xy$ is true in $\mathfrak{A}$ iff there are more things x in A satisfying $(My)\Phi xy$ than things x in A not satisfying $(My)\Phi xy$. That is, the number of a's in A that stand in the relation Φ to more than half the objects in the universe is larger than the number of a's in A for which this does not hold. Formally, if g is an assignment of elements in A to the individual variables of the language, then $\mathfrak{A} \models (Mx)(My)\Phi(x, y)[g]$ iff there are more elements $a \in A$ for which (i) holds than elements $b \in A$ for which (ii) holds:

(i) There are more elements $c \in A$ for which $\mathfrak{A} \models \Phi(x, y)[g(x/a)(y/c)]$ than elements $d \in A$ for which $\mathfrak{A} \models \sim\Phi(x, y)[g(x/a)(y/d)]$.

(ii) There are at least as many $d \in A$ for which $\mathfrak{A} \models \sim\Phi(x, y)[g(x/b)(y/d)]$ as $c \in A$ for which $\mathfrak{A} \models \Phi(x, y)[g(x/b)(y/c)]$.

19. This is example (3.a) of Barwise and Cooper 1981, p. 160.

20. See, for example, Rescher 1968, pp. 171–172.

21. See Barwise and Cooper 1981, pp. 214–216.

22. Rescher 1962, p. 374.

23. Barwise and Cooper 1981, p. 162.

24. Barwise and Cooper 1981, p. 161.

25. I am using a slightly different notation from that of Barwise and Cooper.

26. Barwise and Cooper 1981, p. 177.

27. Barwise and Cooper 1981, pp. 165–166. Sentences relabeled.

28. Barwise and Cooper 1981, p. 164.

29. Barwise and Cooper 1981, p. 170. In later literature the property of living on is often called "conservativity," after Keenan and Stavi (1986).

30. Barwise and Cooper 1981, pp. 178–179.

31. Barwise and Cooper 1981, p. 179.

32. Barwise and Cooper 1981, p. 170.

33. The limitations of living on are also discussed by such authors as Thijsse (1983, pp. 22–26), van Benthem (1983a, p. 452), and Westerståhl (1989, pp. 28–37). These authors do accept the living-on constraint, though on grounds extraneous to the logico-philosophical principles that guide me here. In conversation Richard Larson suggested to me that *living on* is criterial for the class of *determiners* (identified by their distributional behavior in NPs), while *permutation* is criterial for the class of *quantifiers*. This suggestion is reflected in my remark that Barwise and Cooper may have identified a linguistically significant category of expressions, but this category is not that of *quantifiers*. Indeed, Keenan and Stavi (1986), who were co-originators of the conservativity universal, directed their attention to "determiners" rather than "quantifiers." As for the identification of quantifiers with noun phrases, this has not been widely accepted. Thus van Benthem (1983a) and Westerståhl (1989), for example, treat determiners rather than noun phrases as representing quantifiers in natural language. Furthermore, both Keenan (1987) and van Benthem (1989) allow new, more complex types of natural-language quantifiers that do not coincide either with noun phrases or with determiners. (See also chapter 4 below.)

34. See "Noun Phrases, Generalized Quantifiers, and Anaphora" (1987). Barwise's reasons for rejecting the analysis of proper names as quantifiers have to do with the nature of proper names, though, rather than with the nature of quantifiers.

35. May 1991, p. 353.

Chapter 3

1. Tharp 1975, p. 5. The italics are mine.

2. Vaught 1974, p. 161.

3. See articles 3–6, 8–10, 12, and 14 in Tarski 1983, especially pp. 30, 36–37, 38–40, 60–63, 69–72, 166, 281, 285, 298, and 342.

4. Tarski 1936a, pp. 412–413.

5. Tarski 1936b, p. 401.

6. Tarski 1936a, pp. 414–415.

7. Tarski 1936a, p. 417.

8. In "Truth in a Structure" W. Hodges speculates that Tarski did not talk explicitly about variability of universes in "On the Concept of Logical Consequence," because this paper was intended for a philosophical audience, which, Tarski thought, might not appreciate the point. See Hodges 1986, p. 138.

9. Tarski 1936a, p. 417.

10. This proof is superficially similar to another "proof" of Tarski's claim suggested by Etchemendy (1990, chap. 6). However, the proof proposed by Etchemendy leads to a fallacy. I criticize Etchemendy's reconstruction of Tarski's proof in "Did Tarski Commit 'Tarski's Fallacy'?" (1991).

11. Tarski 1936a, pp. 414–415.

12. See Tarski 1936a, pp. 415–416.

13. Tarski 1936a, pp. 418–419.

14. Tarski 1936a, p. 419.

15. This is a reformulation of Tarski's definition in 1936a, p. 419, n. 1.

16. This definition is different in style from the one proposed in chapter 2. There the universal quantifier was construed as a function rather than a set of sets. However, the two definitions are equivalent. In chapter 2, I construed a (1-place predicative) quantifier Q as a function from subsets of the universe to {T, F}. Here I identify Q with the set of all subsets to which the above function gives the value T.

17. Westerståhl (1976, p. 57) points to another case in which a given logical term has different denotations in different models. This is the case of two models with disjoint universes. Thus, in the case of the existential quantifier, its denotation in a model with a universe of dogs is different from its denotation in a model with a universe of donkeys.

18. Westerståhl (1976, p. 57) proposes a similar characterization, saying that the interpretation of logical constants in a given model is "fixed in advance."

19. Let me add a note on the relation between structures and models. A structure is a model in the most general sense, i.e., not a model for a particular language but a sequence of a set (thought of as a *universe*), and a series of individuals, subsets, and relations based on this set. More precisely, a structure is a sequence $\mathfrak{A} = \langle A, V_1, \ldots, V_n \rangle$, where $V_1, \ldots, V_n$ are individuals in A, subsets of A, relations on A, and/or functions on A. (If $\mathfrak{A}$ is a model *for a language*, then $V_1, \ldots, V_n$ correspond to primitive symbols of the language [of the right type].) Now, given a logical term C—say a 1-place, second-level predicate over first-level predicates (i.e., a Mostowskian quantifier)—and a model $\mathfrak{A}$ with universe A, the semantic definition of C specifies with respect to every subset B of A whether it satisfies C in $\mathfrak{A}$ or not (formally, whether it is a member of $C_{\mathfrak{A}}$ or not). So in constraining the definition of C, we have to take into account all structures of the type $\langle A, B \rangle$, where A is the universe of some model for the language and B is a subset of A. This kind of structure is considered in condition (E).

20. Lindenbaum and Tarski 1934–1935, p. 385. The formal theorem is, "Every sentence of the form

$$(x', x'', y', y'', z', z'', \ldots, R) :. R \begin{matrix} x', y', z', \ldots \\ x'', y'', z'', \ldots \end{matrix} . \supset :$$

$$\sigma(x', y', z', \ldots) . \equiv . \sigma(x'', y'', z'', \ldots)$$

is logically provable," where

$$R \begin{matrix} x', y', z', \ldots \\ x'', y'', z'', \ldots \end{matrix}$$

expresses "the fact that the relation R maps the class of all individuals onto itself in one-one fashion, so that the individuals, classes, relations etc., $x', y', z', \ldots$ are mapped on $x'', y'', z'', \ldots$, respectively and "$\sigma(x, y, z, \ldots)$" is a general scheme of a sentential function with no extralogical constants and with the variables $x, y, z, \ldots$ free (p. 385).

21. Mautner 1946, p. 345. The citation has italics removed.

22. See Mostowski 1957, p. 13, n. 3.

23. For Lindström's theorems, see Lindström 1969 or 1974. A textbook presentation appears in Ebbinghaus, Flum, and Thomas 1984.

24. Tarski 1986, p. 149.

25. See McCarthy 1981, sections 3 to 5.

Chapter 4

1. As in the appendix, when discussing Higginbotham and May's work, I follow their use of "automorphism (of sets)" where I usually use "permutation." I should add that in the context of their investigations Higginbotham and May regard (b.1) as the limit of "true" quantifiers.

2. May 1989, p. 397.

3. I will explain the context very briefly. In "Questions, Quantifiers, and Crossing" (1981), Higginbotham and May explained crossing coreference (as in (98)), using absorption as follows: If Q_1 and Q_2 are two 2-place predicative quantifiers, A is a set, and R, S are relations included in A^2, then Q_1 and Q_2 can be absorbed by the relational quantifier Q_1/Q_2 of type $\langle 2, 2 \rangle$, defined as

$(Q_1/Q_2)_A(R, S) = T$ iff
$Q_1(\{a \in A : a \in \mathrm{Dom}(R)\}, \{a \in A : Q_2(\{b \in A : aRb\}, \{b \in A : aSb\}) = T\}) = T.$

When applied to (95) and (98), this rule yields respectively

(i) $(\forall/\exists\ xy)(Mx\ \&\ Wy, Lxy)$

and

(ii) $(\forall/\exists\ xy)[(Px\ \&\ Sxy)\ \&\ (My\ \&\ Cyx), Hxy]$,

where $\forall/\exists$ is interpreted according to the definition above. Clark and Keenan (1986) show, with a counterexample, that the analysis of (95) by means of (i) is incorrect. Take a nonempty universe A consisting only of men. Intuitively, (95) is false in A. But the relation "$Mx\ \&\ Wy$" is empty in A (there is no pair $\langle a, b \rangle$ such that a is a man and b is a woman), and hence (i) comes out true. Clark and Keenan proceed to investigate several other possibilities of absorption schemas. While their operator (described in the main body of the chapter) is adequate for (95), it fails for (98). On the other hand, a $\langle 2, 2, 2\rangle$ absorption operator that they propose works for (98) but not for (95). As I have shown in the present chapter, the absorption operator has to be of type $\langle 1, 2, 2\rangle$, and the absorption operator I propose accurately represents both (95) and (98). For further discussion of this issue, see May 1990.

4. Clark and Keenan (1986) take the following to be a natural paraphrase of (98): "Every pilot who shot at some Mig that chased him hit some Mig that chased him that he shot at."

5. This sentence is (4.c) in Keenan 1987, p. 110.

6. Boolos 1981, p. 466.

Chapter 5

1. Henkin 1959, pp. 179–180.

2. For simplicity I assume that (1) has no free variables. I make similar assumptions throughout the chapter. I speak of "truth in a model" rather than of "satisfaction by an assignment in a model," and I formulate the definitions as if I were dealing only with sentences. It is easy to extend these formulations to formulas with free variables.

3. For the Skolem normal form theorem, see, for example, Enderton 1972, p. 275.

4. Henkin 1959, p. 181.

5. Walkoe 1970, p. 538.

6. Hintikka 1973, p. 344, sentence (37).

7. Hintikka 1973, p. 344.

8. Hintikka 1973, p. 345, sentence (39).

9. Fauconnier 1975, p. 560, sentence (10).

10. Dummett 1973, pp. 8f.

11. An alternative reading is, q_1 a's in A are such that for each one of them [q_2 b's in A are such that for each one of them [q_3 c's in A are such that for each one of them "$R^3(a, b, c)$" is true in $\mathfrak{A}$]].

12. I wish to thank Robert May for this example.

13. See van Benthem 1989.

14. We can show that (a) through (d) present four distinct notions by describing situations that distinguish between them:

- $A \subseteq \mathrm{Dom}(R)$, $B \subseteq \mathrm{Ran}(R)$, and A and B are properly included in the universe. When $q_1 = q_2 =$ "all," (a), (c), and (d) are true, while (b) is false. Hence (a) $\neq$ (b), (c) $\neq$ (b), (d) $\neq$ (b).
- $A \subset \mathrm{Dom}(R)$, $B \subset \mathrm{Ran}(R)$, where $\subset$ means "is a proper subset." When $q_1 = q_2 =$ "only," (a) and (d) are false, while (c) is true. Hence (a) $\neq$ (c), (d) $\neq$ (c). When q_1 = "all" and q_2 = "only," (a) is false and (d) is true. Hence (a) $\neq$ (d).

15. I would like to thank an anonymous referee of *Linguistics and Philosophy* for suggesting (34) as an example of independent quantification that, unlike (28) to (30), cannot be analyzed as (23).

16. However, to analyze "Mostly women were elected to the vacant seats in Congress" we have to define independent quantification of type (e):

$$\left.\begin{array}{l|l} (Q_1^2 x) & \Psi_1 x, \\ & \\ (Q_2^2 y) & \Psi_2 y, \end{array}\right. \quad \Phi xy =_{\mathrm{df}} (Q_1^2 x)[\Psi_1 x, (\exists y)(\Psi_2 y \ \&\ \Phi xy)] \ \&\ (Q_2^2 y)[\Psi_2 y, (\exists x)\Phi xy].$$

We get the intended reading of "Mostly women . . ." when we construe "mostly" and the plural "the" as 2-place Mostowskian quantifiers defined as follows: "(The^2 x)($P_1 x$, $P_2 x$)" is true in a model $\mathfrak{A}$ iff the extension of "$P_1 x$" in $\mathfrak{A}$ is not empty and the extension of "$P_1 x$ & $\sim P_2 x$" in $\mathfrak{A}$ is empty. "(mostly^2 x)($P_1 x$, $P_2 x$)" is true in $\mathfrak{A}$ iff the extension of "$P_1 x$ & $P_2 x$" in $\mathfrak{A}$ is larger than the extension of "$\sim P_1 x$ & $P_2 x$" in $\mathfrak{A}$. (For a definition of "mostly" by a Mostowskian cardinality function t, see chapter 2, section 5.)

17. Johan van Benthem suggested to me in correspondence that we characterize independent quantifiers as "scope-free," defined as follows: "Q_1 As stand in the relation R to Q_2 Bs" is scope-free iff it satisfies (1) invariance under passive transformations ("Q_1 As stand in the relation R to Q_2 Bs" is logically equivalent to "Q_2 Bs stand in the relation $\breve{R}$ to Q_1 As," where $\breve{R}$ is the converse of R) and (2) domain/range invariance (if S is a relation such that $\mathrm{Dom}(R) = \mathrm{Dom}(S)$ and $\mathrm{Ran}(R) = \mathrm{Ran}(S)$, then "$Q_1$ As stand in the relation R to Q_2 Bs" is logically equivalent to "Q_1 As stand in the relation S to Q_2 Bs"). This definition applies to independent quantifiers of types (a) to (c). Van Benthem suggests that if we have evidence that a natural language sentence of the above form satisfies the two invariance conditions (1) and (2), its logical form is that of independent quantification. See van Benthem 1989.

18. A quantifier Q is monotone-increasing iff $(Qx)\Phi x$ and $(\forall x)(\Phi x \rightarrow \Psi x)$ imply $(Qx)\Psi x$.

19. Barwise 1979, p. 63.

20. Barwise 1979, p. 60, sentences (21) and (22).

21. Q is monotone-decreasing iff $(Qx)\Phi x$ and $(\forall x)(\Psi x \rightarrow \Phi x)$ imply $(Qx)\Psi x$. Q is non-monotone iff it is neither monotone-increasing nor monotone-decreasing.

22. Barwise 1979, p. 64.

23. Barwise 1979, pp. 65–66. Labels changed.

24. Barwise 1979, pp. 62–64.

25. Thus the structure of a maximal consistent set of formulas gives us enough information to construct a syntactic model as in Henkin's proof of the completeness of standard first-order logic. (I wish to thank Charles Parsons for this example.) For the importance of maximality, note, for example, Zorn's lemma and its numerous uses.

26. My italics.

27. Barwise 1979, p. 62, sentence (23). See also (25). My italics.

28. Barwise 1979. My italics.

29. To see that linear quantification is a particular instance of Henkin-Barwise complex quantification, we have to express the conception of branching embedded in (49) more generally so that it applies to any partially ordered quantifier prefix. I will not discuss the nature of such a definition here, but in the case of "$(Q_1x)(Q_2y)\Phi xy$" the definition I have in mind will yield the following second-order counterpart:

$(\exists X)(\exists R)[(Q_1x)Xx$ & X is a maximal set such that
$(\forall x)(Xx \rightarrow (Q_2y)Rxy)$ &

R is a maximal relation such that $(\forall x)(\forall y)(Rxy \rightarrow \Phi xy)]$.

30. The quantifiers over which $\mathcal{Q}_2$ ranges are higher-order Mostowskian quantifiers that treat pairs as single elements.

31. This example was made up before *glasnost*.

32. Another conjecture expressible in terms of the general definition schema was suggested by an anonymous referee of *Linguistics and Philosophy*. Compare the following:

(i) In the class, most of the boys and most of the girls *all* like each other;

(ii) In the class, most of the boys and most of the girls like each other.

The conjecture is that the difference between (i) and (ii) is in the intended inner quantifier-condition. The presence of the explicit "all" in (i) indicates that the inner quantifier-condition is *each all*. However, the absence of "all" in (ii) signifies that there the inner quantifier-condition is weaker. The reviewer suggests that this condition is, however, stronger than *each–some/some–each* (independence). *Each–most* appears appropriate.

33. I did, however, discuss Westerståhl's definition in "Two Approaches to Branching Quantification" (1990a).

34. Westerståhl 1987, p. 274.

Chapter 6

1. Putnam 1966, p. 106.
2. Putnam 1966, p. 106.
3. Russell 1919, pp. 201–202.
4. See Putnam 1967, p. 16.
5. Some of the themes developed in this section regarding the interplay between logic and ontology appear in Charles Parsons "Ontology and Mathematics" (1971a) and "A Plea for Substitutional Quantification" (1971b). The observation that by augmenting one's logic, one can save on ontology was made earlier by Hartry Field in *Science without Numbers: A Defense of Nominalism* (1980), preface and chapter 9.
6. See Etchemendy 1983 and 1990. Although I accept Etchemendy's account and criticism of interpretational semantics, my view of the relation between Tarski semantics and interpretational semantics differs radically from his.
7. See Quine 1970, p. 91.

Appendix

1. I formulate the definitions and the theorems in terms taken from Higginbotham and May 1981. In particular, I follow their use of "(set) automorphism," where before I used "permutation."
2. Higginbotham and May do not include a description of the function h in their formulation of the theorem (nor do they present a proof). I believe my formulation of the theorem is consistent with their intentions.

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Index of Notation

Page numbers indicate the first introduction of a symbol and its definition.

Use of Variables

The list below gives the most common use of different letters, but in some places the same letter is used for different objects. The letters might occur with subscripts or superscripts.

First-Order Logic

Index of Terms