A CONCEPTION OF TARSKIAN LOGIC*

BY

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Which logic is the right logic? In a paper so titled Leslie Tharp[1] poses the question: What properties should a logical system have? In particular: Is standard 1st-order logic the right logic? The question asked in this paper is somewhat less general: Which logic is Tarski's logic? More precisely: Are the basic principles of Tarskian logic exhausted by the standard 1st-order system or does it take a new, extended logic, to fully realize them? (By 'Tarskian logic' I here understand the modern semantic conception of logic as it evolved out of Tarski's theory.) To answer questions on the adequacy of a system of logic, Tharp says, it is essential that we acquire first an idea of "the role logic is expected to play."[2] I think Tharp's point is important, and with this guideline in mind I will turn to Tarski's early work on the foundations of semantics.[3]

I. *The Task of Logic, the Origins of Semantics*

In "The Concept of Truth in Formalized Languages", "On the Concept of Logical Consequence" and other writings[4] Tarski presents logical semantics as providing (i) a definition of the general concept of truth for formalized languages, and (ii) definitions of the logical concepts 'logical truth', 'logical consequence', 'consistency', etc., for such languages.

The main purpose of (i) is to secure metalogic against semantic paradoxes. Tarski worried lest the uncritical use of semantic concepts prior to his work concealed an inconsistency: a hidden fallacy would undermine the entire venture. He therefore sought precise, materially as well as formally correct, definitions for 'truth' and related notions which would serve as a hedge against paradox. This aspect of Tarski's work is well known. In "Model Theory Before 1945", Robert Vaught[5] puts Tarski's enterprise in a slightly different light: Work in model theory

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before the development of formal semantics was based on an intuitive notion of truth which was not defined mathematically. "This meant that the theory of models (and hence much of metalogic) was indeed not part of mathematics . . . . [Tarski's] major contribution was to show that the notion 'σ is true in $\mathscr{A}$' can simply be *defined* inside of ordinary mathematics, for example, in ZF."[6]

On both accounts the motivation for (i) has to do with the adequacy of the system designed to carry out the logical project, not with the logical project itself. The goal of logic is not the mathematical definition of 'true sentence', and (i) is, therefore, a secondary, if crucially important, task of Tarskian logic. (ii), on the other hand, does reflect Tarski's vision of the role of logic. In paper after paper throughout the early 30's, Tarski describes the logical project as follows:[7] The goal is to study the properties of *deductive systems*. A *(closed) deductive system* is the set of all sentences which follow logically from a set X of sentences (of a given formal language), namely, a *formal theory* in contemporary terminology. The task of logic is twofold: (A) the construction of a logical framework for formal (formalized) theories; (B) the investigation of the logical properties of formal theories relative to the logical framework constructed in (A). (Note that the logical framework itself can be viewed as a deductive system, namely, by taking X to be the set of logical axioms.) (A) determines the enterprise of logic proper; (B)—the enterprise of metalogic. The concept of *logical consequence* is the key concept of metalogic according to Tarski. Once the definition of 'logical consequence' is given, we can easily obtain not only the notion of a formal theory (deductive system) but also those of a logically true sentence, logically equivalent sets of sentences, axiomatizability, completeness and consistency of a set of sentences.

Whence semantics? Prior to Tarski's "On The Concept of Logical Consequence", the definitions of 'logical consequence' and the other logical concepts were proof-theoretical, i.e., in terms of logical axioms and rules of inference. Thus, given a formal system $\mathscr{L}$ which includes, in addition to a formal language, a set of logical axioms, A, and a set of rules of inference, R—the set of *logical consequences* of a sentence X in $\mathscr{L}$ was defined as the smallest set of sentences of $\mathscr{L}$ which includes X and the axioms in A, and is closed under the rules in R. The need for semantic definitions of the same concepts arose when Tarski realized that there was a serious gap between the proof-theoretical definitions and the intuitive concepts they were intended to capture: many intuitive consequences of formal theories were undetectable by the standard system of proof. Thus, the sentence 'For every natural number n, Pn' seems to follow, in some important sense, from the set of sentences 'Pn', where n is a natural number, but there is no way to express this fact by the proof method for standard 1st-order logic.[8] This situation, Tarski said, shows that proof theory, by itself, is inadequate for the task of metalogic. One might contemplate extending the system by adding new rules of inference, but to no avail. Gödel's discovery of the incompleteness of the deductive system of Peano Arithmetic showed that:

In every deductive theory (apart from certain theories of a particularly elementary nature), however much we supplement the ordinary rules of inference by new purely structural rules, it is possible to construct sentences which follow, in the usual sense, from the theorems of this theory, but which nevertheless cannot be proved in this theory on the basis of the accepted rules of inference.[9]

Tarski's conclusion was that proof theory can provide only a partial account of the metalogical concepts. A new method is called for, which will permit a more comprehensive systematization of the intuitive content of these concepts.

Our pretheoretical understanding of the logical concepts is based, according to Tarski, on certain intuitions of the relationship between linguistic expressions and the objects they refer to (the situations they describe). The discipline which studies relations of this kind is *semantics*.

We . . . understand by semantics the totality of considerations concerning those concepts which, roughly speaking, express certain connexions between the expressions of a language and the objects and states of affairs referred to by these expressions.[10]

The precise formulation of the intuitive content of the logical concepts is, therefore, a job for semantics. (Although the relation between the set of sentences 'Pn' and the universal quantification '(∀x)Px', where x ranges over the natural numbers and 'n' stands for a name of a natural number, is not logical consequence, we will be able to characterize it accurately within the framework of Tarskian semantics, e.g., in terms of ω-models.)

II. The Semantic Definition of 'Logical Consequence' and the Emergence of Models

Tarski describes the intuitive content of the concept 'logical consequence' as follows:

Certain considerations of an intuitive nature will form our starting-point. Consider any class K of sentences and a sentence X which follows from the sentences of this class. From an intuitive standpoint it can never happen that both the class K consists only of true sentences and the sentence X is false. Moreover, . . . we are concerned here with the concept of logical, i.e. *formal*, consequence, and thus with a relation which is to be uniquely determined by the form of the sentences between which it holds . . . The two circumstances just indicated . . . seem to be very characteristic and essential for the proper concept of consequence.[11]

We can express the two conditions set by Tarski on a correct definition of 'logical consequence' by (C1) and (C2) below:

(C1) If X is a logical consequence of K, then X is a *necessary* consequence of K in the following intuitive sense: it is impossible that all the sentences of K are true and X is false.
(C2) Not all cases of intuitive consequence fall under the concept of logical consequence: only those in which the consequence relation between a set of sentences K and a sentence X is based on *formal* relationships between the sentences of K and X do.

In order to give a definition of 'logical consequence' based on (C1) and (C2), Tarski introduces the notion of *model*. In current terminology, given a formal system $\mathscr{L}$, a *model for* $\mathscr{L}$ is a pair, $\mathscr{A} = <D,A>$, where A is a set and D is a function which assigns the nonlogical primitive constants of $\mathscr{L}$, $t_1,t_2, \ldots$, elements (or constructs of elements) in A: if t_i is an individual constant, $D(t_i)$ is a member of A; if t_i is an n-place 1st-order predicate, $D(t_i)$ is an n-place relation included in A^n; etc. We will say that the function D assigns to $t_1,t_2, \ldots$ denotations in A. Any pair of a set A and a denotation function D determines a model for $\mathscr{L}$. Given a theory $\mathscr{T}$ in a formal system $\mathscr{L}$, we say that a model $\mathscr{A}$ for $\mathscr{L}$ is a *model of* $\mathscr{T}$ iff (if and only if) every sentence of $\mathscr{T}$ is true in $\mathscr{A}$. Similarly, $\mathscr{A}$ is a model of a sentence X of $\mathscr{L}$ iff X is true in $\mathscr{A}$.) The definition of 'the sentence X is true in a model $\mathscr{A}$ for $\mathscr{L}$' is given in terms of satisfaction: *X is true in* $\mathscr{A}$ iff every assignment of elements in A to the variables of $\mathscr{L}$ satisfies X in $\mathscr{A}$. The definition of satisfaction is based on "The Concept of Truth in Formalized Languages." I assume the reader is familiar with this definition.

The formal definition of 'logical consequence' in terms of models proposed by Tarski is:

(LC) The sentence X *follows logically* from the sentences of the class K iff every model of the class K is also a model of the sentence X.[12]

The definition of 'logical truth' immediately follows:

(LT) The sentence X (of $\mathscr{L}$) is *logically true* iff every model (for $\mathscr{L}$) is a model of X.

(A historical remark: Some philosophers claim that Tarski's 1936 definition of a model is essentially different from the one currently used because in 1936 Tarski did not require that models vary with respect to their universes. This issue does not really concern us here since what we are interested in is the legacy of Tarski, not this or that historical stage in the development of his thought. For the intuitive ideas we go to the early articles where they are most explicit, while the formal constructions are those which appear in his mature work.

Notwithstanding the above, it seems to me highly unlikely that in 1936 Tarski intended all models to share the same universe. This is because such a notion of model is incompatible with the most important model-theoretic results obtained by that time. Thus, the 1915–1920/22–1927/28 Löwenheim-Skolem-Tarski theorem says that if a 1st-order theory has a model with an infinite universe, it has a model with a universe of cardinality α for every infinite α. Obviously, this theorem does not hold if one universe is common to all models. Similarly, Gödel's 1930 completeness theorem fails: If all models share the same universe, then for every positive integer n, one of the two 1st-order statements, 'there are more than n things' and 'there are at most n things', is true in all models, hence, according to (LT), it is logically true. But no such statements are provable from the logical axioms of standard 1st-order logic. Be that as it may, the Tarskian concept of a model discussed here does include the requirement that any nonempty set is the universe of some model for the given language.)

Does (LC) satisfy the intuitive requirements on a correct definition of 'logical consequence' given by (C1) and (C2) above? According to Tarski it does:

> It seems to me that everyone who understands the content of the above definition must admit that it agrees quite well with common usage. . . . it can be proved, on the basis of this definition, that every consequence of true sentences must be true, and also that the consequence relation which holds between given sentences is completely independent of the sense of the extra-logical constants which occur in these sentences.[13]

In what way does (LC) satisfy (C1)? Tarski mentions the existence of a proof but does not provide a reference. John Etchemendy[14] proposes a very simple argument which, I believe, is in the spirit of Tarski:

(PR) Assume X is a logical consequence of K, i.e., X is true in all models in which all the members of K are true. Suppose that X is not a necessary consequence of K. Then it is possible that all the members of K are true and X is false. But in that case there is a model in which all the members of K come out true and X comes out false. Contradiction.

The argument is simple. However, it is based on a crucial assumption:

(AS) If K is a set of sentences and X is a sentence (of a Tarskian language) such that it is intuitively possible for all the members of K to be true while X is false, then there is a (Tarskian) model in which all the members of K come out true and X comes out false.

(AS) is equivalent to the requirement that, given a logic $\mathscr{L}$, every possible state of affairs relative to the expressive power of $\mathscr{L}$ be represented by some model for $\mathscr{L}$. (Note that (AS) does not entail that every state of affairs represented by a model for $\mathscr{L}$ is possible. This is in accordance with Tarski's view that the notion of logical possibility is weaker than—hence, different from—the general notion of possibility (see (C2)).) Is the assumption (AS) fulfilled by Tarski's model-theoretic semantics?

We can show that (AS) holds at least for standard 1st-order models. Let $\mathscr{L}$ be a 1st-order system, K—a set of sentences of $\mathscr{L}$, and X—a sentence of $\mathscr{L}$. Suppose it is intuitively possible that all the members of K are true and X is false. Then, presuming the rules of inference of standard 1st-order logic are necessarily truth-preserving, $K \cup \{\sim X\}$ is intuitively consistent in the proof-theoretic sense: for no 1st-order sentence Y, both Y and $\sim Y$ are provable from $K \cup \{\sim X\}$. It follows from the Completeness Theorem for 1st-order logic that there is a model for $\mathscr{L}$ in which all the sentences of K are true and X is false.

As for (C2), in standard Tarskian logic, logical consequences depend only on the definitions of the truth-functional connectives, the existential (universal) quantifier and identity. Generally speaking, the mathematical nature of these operators ensures the satisfaction of (C2). More particularly, Tarski took the fact that the relation of logical consequence defined by (LC) "is completely independent of the sense of the extra-logical constants" as a mark of *formality*. This characterization is elucidated in a 1935 article co-authored by Lindenbaum: "Every relation between objects (individuals, classes, relations) which can be expressed by purely logical means is invariant with respect to every one-one mapping of the 'world' (i.e., the class of all individuals) onto itself."[15] In particular, the logical terms themselves are invariant under such mappings. I.e., given a model $\mathscr{A}$ with a universe A and a 1-place formula 'Φx', '$(\forall x)\Phi x$' is true in $\mathscr{A}$ iff for any 1-place formula 'Ψx' whose extension in $\mathscr{A}$ is obtained from that of 'Φx' by some permutation of A, '$(\forall x)\Psi x$' is true in $\mathscr{A}$. Similarly for identity and the truth-functional connectives. (We can think of 'truth' and 'falsity' as denoting the universal and empty set, respectively.) Borrowing a form of speech from A. Mostowski,[16] we can say that the standard logical operators are formal in not distinguishing between different elements in the universe of a given model. Thus, if X is a logical consequence of K, this relationship has nothing to do with the objects referred to by X and the sentences in K, but only with a certain formal pattern of objects-possessing-properties (standing-in-relations) which persists through all models for the language.

We see that the logical consequences of standard 1st-order logic satisfy Tarski's intuitive requirements due to two facts: every possible state of affairs relative to a given 1st-order language is represented by some model for the language, and all the standard logical terms are formal in the sense indicated above. An example of a consequence which is not logical according to Tarski's definition would be 'b is red all over; therefore b is not blue all over'. It is easy to see that this consequence does not satisfy (C2).

I think the conditions (C1) and (C2) on the key concept of logical consequence delineate the scope as well as the limit of Tarski's enterprise: the development of a conceptual system in which the concept of logical consequence ranges over *all* and *only* intuitively *necessary-and-formal* consequences. (Since our intuitions leave some consequences undetermined with respect to necessity and formality, the boundary of the enterprise is: The system should be so defined as to exclude consequences which are intuitively definitely not necessary-and-formal. This leaves a gray area which is to be dealt with based on other considerations.)

We have seen that Tarskian semantics, applied to standard 1st-order logic, classifies as logical only consequences which are both necessary and formal. We now ask: Does standard 1st-order logic suffice to yield *all* the necessary and formal consequences with a 1st-order (extensional) vocabulary? Could not the standard system be extended so that new consequences, satisfying the intuitive conditions but undetected within the standard system, fall under Tarski's definition? Tarski himself all but asked the same question. He ended "On The Concept of Logical Consequence" with the following note:

> Underlying our whole construction is the division of all terms of the language discussed into logical and extra-logical. This division is certainly not quite arbitrary. If, for example, we were to include among the extra-logical signs the implication sign, or the universal quantifier, then our definition of the concept of consequence would lead to results which obviously contradict ordinary usage. On the other hand no objective grounds are known to me which permit us to draw a sharp boundary between the two groups of terms. It seems to be possible to include among logical terms some which are usually regarded by logicians as extra-logical without running into consequences which stand in sharp contrast to ordinary usage. . . .[17]

The question 'What is the full scope of Tarskian logic?' we will ask in the form: What is the widest notion of *logical term* for which the

Tarskian definition of 'logical consequence' gives results compatible with (C1) and (C2)? Tarski doubted that a general criterion for logical terms will ever be found.[18] I think his doubt is unjustified: although a definition in terms of (C1) and (C2) would be circular, (C1) and (C2) can be used as a measure for an independent criterion. Below I will try to show that Tarski's own work naturally leads to such a criterion.

III. Logical and Extra-Logical Terms: Role in a System

What makes a term logical/extra-logical in Tarski's system? Considering the question from the 'functional' point of view suggested by Tharp, we ask: How does the dual system of a formal language and its model-theoretic semantics accomplish the task of logic? In particular, what is the role of logical and extra-logical constants in determining logical truths and consequences?

A. *Extra-Logical Constants.* Consider the statement:

(1) Some horses are white,

formalized in standard 1st-order logic by:

(2) $(\exists x)(Hx \ \& \ Wx)$.

How does Tarski succeed in giving this statement truth conditions which render it logically indeterminate (i.e., neither logically true nor logically false), in accordance with our clear pretheoretical intuitions? The crucial point is that the common noun 'horse' and the adjective 'white' are interpreted within models in such a way that their intersection is empty in some models, not empty in others. Similarly, for any natural number n, the sentence

(3) There are n white horses

is logically indeterminate because in some, but not all, models 'horse' and 'white' are so interpreted as to make their intersection of cardinality n. A similar configuration would make

(4) Finitely many horses are white

also logically indeterminate, provided we could express 'finitely many' in the logic.

In short, what is special to extra-logical terms like 'horse' and 'white' in Tarskian logic is their *strong semantic variability*. Extra-logical terms have no independent meaning: they are interpreted only *within* models. Their meaning in a given model is nothing more than the value the denotation function D assigns them in that model. We cannot speak about *the* meaning of an extra-logical term: being extra-logical implies that nothing is ruled out with respect to such a term. Hence, the totality of interpretations of any given extra-logical term in the class of all models for the formal system is exactly the same as that of any other extra-logical term of the same syntactic category. Given any set of objects, any possible denotation of the extra-logical terms in this set of objects (in accordance with their syntactic category) is represented by some model. Since every set of objects is the universe of some model, any possible state of affairs—configuration of individuals, properties, relations and functions—vis-à-vis the extra-logical terms of a given formalized language (possible, that is, with respect to their meaning prior to formalization) is represented by some model.

Formally, we can define Tarskian extra-logical terms as follows:

(TET) $\{e_1, e_2, \ldots\}$ is the set of primitive *extra-logical terms* of a Tarskian logic $\mathcal{L}$ iff for every set A and every function D which assigns to $e_1, e_2, \ldots$ denotations in A (in accordance with their syntactic categories), there is a model $\mathcal{A}$ for $\mathcal{L}$ such that $\mathcal{A} = \langle D, A \rangle$.

It follows from (TET) that primitive extra-logical terms are semantically unrelated to one another. As a result, complex extra-logical terms, produced by intersections, unions, etc. of primitive extra-logical terms (e.g., 'horse and white') are strongly variable as well.

Note: It is essential to take into account the strong variability of extra-logical terms in order to understand the meaning of various claims of logicality. Consider, for instance, the statement

(5) $(\exists x)(x = \text{Jean-Paul-Sartre})$

which is logically true in a Tarskian logic with 'Jean-Paul-Sartre' as an extra-logical individual constant. Does the claim that (5) is logically true mean that the existence (unspecified with respect to time) of the deceased French philosopher Jean-Paul Sartre is a matter of logic? Obviously not. The logical truth of (5) reflects the principle that if a term is used in a language to name objects, then in every model for the language some object is named by that term. But, since 'Jean-Paul Sartre' is a strongly variable term, what (5) says is 'There is *a* Jean-Paul Sartre', not '[*The* French philosopher] Jean-Paul Sartre exists'.

B. *Logical Constants*. It has been said that to be a logical constant in a Tarskian logic is to have *the same* interpretation in all models. Thus, for 'red' to be a logical constant in a logic $\mathscr{L}$, it has to have a constant interpretation in all the models for $\mathscr{L}$. I think this characterization is faulty because it is vague. How do you interpret 'red' *in the same way* in all models? 'In the same way' in what sense? Do you require that in every model there be the same number of objects falling under 'red'? But for every number other than 1 there is a model which cannot satisfy this requirement simply because it does not have enough elements. So at least in some way—i.e., cardinality-wise—the interpretation of 'red' must vary from model to model.

The same thing holds for the standard logical constants of Tarskian logic. Take the universal quantifier: In every model for a 1st-order logic $\mathscr{L}$ the universal quantifier is interpreted as a singleton set. But in a model with 10 elements it is a set of a set with 10 elements, whereas in a model with 9 elements it is a set of a set with 9 elements. Are these interpretations the same?

I think that what distinguishes logical constants in Tarski's semantics is not the fact that their interpretation does not vary from model to model (it does!), but the fact that they are interpreted *outside* the system of models. The meaning of a logical constant is not given by the definitions of particular models, but is part of the same metatheoretical machinery which is used to define models. The meaning of logical constants is given by *rules external to the system*, and it is due to the existence of such rules that Tarski could give his inductive definition of truth (satisfaction) for well-formed formulas of any given language of the logic. Syntactically, the logical constants are 'fixed parameters' in the inductive definition of the set of well-formed formulas; semantically, the rules for the logical constants are the functions on which the definition of satisfaction by recursion (on the inductive structure of the set of well-formed formulas) is based.

How would different choices of logical terms affect the extension of 'logical consequence'? Well, if we contract the standard set of logical terms, some consequences which are intuitively necessary-and-formal (namely, logical consequences of standard 1st-order logic) will become nonlogical. If, on the other hand, we take any term whatsoever as logical, we will end up with new 'logical' consequences which are intuitively not necessary-and-formal. The first case does not require further elaboration. As for the second case, take, for instance, the natural-language terms 'Jean-Paul Sartre' and 'accepted the Nobel Prize in literature' and suppose we use them as logical terms in a Tarskian logic by keeping their usual denotation 'fixed'. I.e., the semantic counterpart of 'Jean-Paul Sartre' will be the existentialist French philosopher Jean-Paul Sartre, and the semantic counterpart of 'accepted the Nobel Prize in literature' will be the set of all actual persons up to the present who accepted the Nobel Prize in literature. Then

(6) Jean-Paul Sartre accepted the Nobel Prize in literature

will come out false, according to Tarski's rules of truth (satisfaction), no matter what model we are considering. (This is because, when determining the truth of (6) in any given model $\mathcal{A}$ for the logic, we do not have to look in $\mathcal{A}$, but instead, we need to examine two fixed entities outside the apparatus of models and determine whether the one is a member of the other.) This renders (6) logically false, and any sentence of the language we are considering follows logically from it according to Tarski's definition, in contradiction with the pretheoretical conditions (C1) and (C2).

The case described above violates two principles of Tarskian semantics: (i) 'Jean-Paul Sartre' and 'accepted the Nobel Prize in literature' are not formal; (ii) the truth conditions for (6) bypass the very device which serves, in Tarskian semantics, to distinguish material from logical consequences, namely—the apparatus of models. No wonder the definition of 'logical consequence' fails.

It is easy to see that each of the above violations by itself suffices to undermine Tarski's definition. (i) is obvious. As for (ii), suppose we define a logical term formally but without reference to models. Say, we interpret the universal quantifier as referring to some fixed 'universe' defined in a formal manner (e.g., the universe of natural numbers). Then, a statement like

(7) Every object is different from at least three objects

will turn out logically true (assuming the 'universal' set has at least 4 members). But this result is obviously in disagreement with the condition (C1).

Tarski himself said that in the extreme circumstance in which all terms of the language are construed as logical, the concept of logical consequence will coincide with that of material consequence.[19] Unlike 'logical consequence', the concept of material consequence is defined without reference to models:

(MC) The sentence X is a *material consequence* of the sentences of the class K iff at least one sentence of K is false or X is true.[20]

I think Tarski's last claim is inaccurate: By adding new logical constants we can distort the distinction between logical and material truths and consequences, but we cannot eliminate it altogether. This we can see as follows: Let $\mathscr{L}$ be a 1st-order logic with any new logical terms in addition to the standard ones. Consider the sentence

(8) There is exactly one thing

or, formally, '$(\exists x)(\forall y)x = y$'. Although (8) is materially false, it is not logically false in the new logic. This is because for each cardinality α, there is a model for $\mathscr{L}$ with a universe of cardinality α. Thus, in particular, $\mathscr{L}$ has a model with a universe of cardinality 1. In this model (8) turns out true. The fact that (8) is not logically false is invariant under expansions of the set of logical terms. The case for logical vs. material consequence is basically the same.

So Tarski conceded too much: No addition of new logical terms can trivialize his definitions altogether. Tarski's model-theoretic semantics has a built-in barrier which prevents a complete mergence of logical and material consequence. To turn all material consequences of a given language into logical consequences requires relinquishing the apparatus of models. But this apparatus is at the heart of Tarski's semantics.

We can now see how Tarski's method allows us to construct

(9) Everything is identical with itself

as a logical truth in agreement with our intuitions. The crucial point is that the intuitive meanings of 'is identical with' and 'everything' can be expressed by formal rules that determine, for any given model, which pairs of objects are 'identical' and which subsets of the universe constitute 'everything' in that universe. These rules make (9) true in every model. (Note that if either 'everything' or 'is identical with' received a strongly variable interpretation, (9) would not have been logically true.)

We saw that Tarski's conception of logic does not allow any arbitrary rule to be the semantic definition of a logical constant. At the same time, as Tarski himself indicated, there are many terms other than the standard logical constants which could be construed as logical without violating the intuitive requirements. Consider, for instance, the 2nd-order predicate 'finitely many'. It appears, at least prima facie, that consequences based on this predicate—e.g., 'Exactly one French philosopher refused the Nobel Prize in literature; therefore, finitely many French philosophers did'—are both necessary and formal in Tarski's sense.

Below I will present a definition of 'logical term' which yields new logical consequences in accordance with Tarski's pretheoretical conditions. Among these is the consequence mentioned above. This definition is a natural outcome of our considerations on the nature and purpose of Tarskian logic.

To recapitulate: The conditions (C1) and (C2) require (i) that every possible state of affairs vis-à-vis a given language be represented by some model for the language, and (ii) that logical terms represent formal features of possible states of affairs, i.e., formal properties of (relations among) constituents of states of affairs. To satisfy these requirements the Tarskian logician constructs a dual system, each member of which is itself a complex, syntactic-semantic structure. Of the two subsystems, one includes the extra-logical vocabulary (syntax) and the apparatus of models (semantics). We will call this the *base* of the logic. (Note that only extra-logical terms—not logical terms—play a role in constructing models.) In standard 1st-order logic the base is strictly 1st-order: syntactically, the extra-logical vocabulary includes only singular terms or terms whose arguments are singular; semantically, in any given model the extra-logical terms are assigned only individuals or properties (relations, functions) of individuals.

The second subsystem includes the logical terms and their semantic definitions. Its task is to introduce formal structure into the logic. Syntactically, logical terms are formula-building operators; semantically, they are assigned pre-fixed functions on models which express formal properties of, relations among, and functions of 'elements of models' (objects in the universe and constructs of these). Since logical terms are meant to represent formal properties of elements of models corresponding to the extra-logical vocabulary, their order is generally higher than that of nonlogical terms. Thus, in standard 1st-order logic identity is the only strictly 1st-order logical term. The universal and existential quantifiers are 2nd-order, semantically as well as syntactically, and the logical connectives, too, are not strictly 1st-order (in the sense indicated above). As for singular terms, these can never be construed as logical. This is because singular terms represent atomic components of models, and atomic components, being atomic, have no structure (formal or informal). We will say that the system of logical terms constitutes a *superstructure* for the logic.

The two subsystems are combined by imposing the logical superstructure upon the nonlogical base. Syntactically, this is done by rules for forming well-formed formulas, and semantically—by rules for determining truth (satisfaction) in a model.

Now, to satisfy the conditions (C1) and (C2) it is essential that no logical term represent a property or a relation which is intuitively variable from one state of affairs to another. It is also essential that logical terms be assigned formal denotations. Finally, it is essential that logical terms be defined over models—all models—so that all possible states of affairs are taken into account in determining logical truths and consequences.

It appears that if we can specify a series of conditions satisfied only by properties, relations and functions as described above, we will have succeeded in defining 'logical term' in accordance with Tarski's basic principles. In particular, the Tarskian definition of 'logical consequence' (and the other metalogical concepts) will give correct results, in agreement with (C1) and (C2).

IV. Definition of Logical Terms for Tarskian Logics

Preliminary remarks: We will not bother to specify conditions on Tarskian truth-functional connectives, because the problem of identifying all the truth-functional connectives that there are has already been solved and the solution clearly satisfies Tarski's requirements. (The standard logic of truth-functional connectives has a base that consists, syntactically, of sentential letters which are extra-logical, and semantically, of a list of all possible assignments of truth-values to these letters. Any possible state of affairs vis-à-vis the language is represented by some assignment. The logical super-structure includes the truth-functional connectives and their semantic definitions. The connectives are, both syntactically and semantically, of a higher level than the sentential letters. Their semantic definitions are pre-fixed: logical connectives are defined by Boolean functions which are formal, and whose arguments and values are the kinds of things which represent possible states of affairs, i.e., truth values and sequences of these. This structure ensures that truths and consequences which hold in all 'models'—i.e., under any assignment of truth values to the sentential letters—are necessary and formal in Tarski's sense.)

As for modal operators, they, too, are outside the scope of our investigation, though for different reasons. First, our definition is based on analysis of the model-theoretic apparatus of Tarskian semantics, and this apparatus is inadequate for the modals. Second, we cannot take it for granted that the task of modal logic is the same as that of 'mathematical' logic. To determine the scope of modal logic and characterize its operators, we need an independent study of its goals and principles.

With respect to the Tarskian conception of 1st-order (mathematical) logic the question yet to be answered is: What are all the 1st- and 2nd-order predicates and functions which can serve as logical terms in that logic? The conditions on logical terms below are intended to answer this question. In formulating these conditions I placed a higher value on clarity of ideas than on economy. As a result the conditions are not mutually independent.

CONDITIONS ON LOGICAL CONSTANTS (OTHER THAN TRUTH-FUNCTIONAL CONNECTIVES) FOR TARSKIAN 1ST-ORDER LOGICS:

(A) A logical constant, C, is, syntactically, an n-place predicate or functor of order 1 or 2, n being a positive integer.

(B) A logical constant, C, is defined by a single, extensional function on models and is identified with its extension.

(C) A logical constant, C, is assigned, in any model (over which it is defined), a denotation which is a construct of elements in the model corresponding to its syntactic category. More specifically, we require that C be defined by a function, f_C, such that given a model $\mathcal{A}$ (with universe A) in its domain:

(a) If C is a 1st-order n-place predicate, then $f_C(\mathcal{A})$ is a subset of A^n.

(b) If C is a 1st-order n-place functor, then $f_C(\mathcal{A})$ is a function from A^n into A.

(c) If C is a 2nd-order n-place predicate, then $f_C(\mathcal{A})$ is a subset of $B_1 \times \ldots \times B_n$, where for $1 \leq i \leq n$:

$$B_i = \begin{cases} A & \text{if i(C) is an individual,} \\ P(A^m) & \text{if i(C) is an m-place predicate.} \end{cases}$$

(i(C) is the i-th argument of C. 'P' stands for 'power-set'.)

(d) If C is a 2nd-order n-place functor, then $f_C(\mathcal{A})$ is a function from $B_1 \times \ldots \times B_n$ into B_{n+1}, where for $1 \leq i \leq n+1$, B_i is defined as in (c).

(D) A logical constant, C, is defined over all models.

(E) A logical constant, C, is defined by a function, f_C, which is invariant under isomorphic structures. I.e.:

(a) If C is a 1st-order n-place predicate, $\mathcal{A}$ and $\mathcal{A}'$ are models with universes A and A′ respectively,
$<b_1,\ldots,b_n> \in A^n$, $<b'_1,\ldots,b'_n> \in A'^n$ and the structures $<A, <b_1,\ldots,b_n>>$, $<A', <b'_1,\ldots,b'_n>>$ are isomorphic, then $<b_1,\ldots,b_n> \in f_C(\mathcal{A})$ iff $<b'_1,\ldots,b'_n> \in f_C(\mathcal{A}')$.

(b) If C is a 2nd-order n-place predicate, $\mathcal{A}$ and $\mathcal{A}'$ are models with universes A and A′ respectively,
$<D_1,\ldots,D_n> \in B_1 \times \ldots \times B_n$, $<D'_1,\ldots,D'_n> \in B'_1 \times \ldots \times B'_n$ (where for $1 \leq i \leq n$, B_i and B'_i are as in (C)(c)), and the structures $<A, <D_1,\ldots,D_n>>$, $<A', <D'_1,\ldots,D'_n>>$ are isomorphic, then:
$<D_1,\ldots,D_n> \in f_C(\mathcal{A})$ iff $<D'_1,\ldots,D'_n> \in f_C(\mathcal{A}')$.

(c) Analogously for functors.

Remarks:

Condition (A) reflects our conception of logical terms as structural components of the language. In particular, it rules out individual constants as logical terms. Note, however, that although an individual by itself cannot be represented by a logical term (since it lacks "inner" structure), it can combine with functions, sets or relations to form a structure which is representable by a logical term. Thus, in the examples below we define a logical constant which represents the structure of the natural numbers with their ordering relation and zero (taken as an individual).

Condition (B) ensures that logical terms are *rigid*. Each logical term has a *prefixed* meaning in the meta-language. This meaning is unchangeable, and is completely exhausted by its model-theoretic definition. That is to say, from the point of view of Tarskian logic, there are no 'possible worlds' of logical terms. Thus, qua logical terms, expressions like 'the number of planets' and '9' are indistinguishable. If you want to express the intuition that the number of planets changes from one possible 'world' to another, construe this expression as an extra-logical term. If, however, you are using it as a logical term (or in the definition of a logical term) you are treating it merely as a 'tag', a synonym of '9'. By requiring that logical terms be defined by fixed functions over models we allow them to represent 'fixed' parameters of changeable situations.

(C) makes sure that logical terms do not *bypass* the apparatus of models. At the same time, it takes care of the correspondence in categories between the syntax and the semantics.

(D) ascertains that *all* possible states of affairs are taken into account in determining logical truths and consequences.

The conditions (B)–(D) express the requirement that logical terms are, semantically, superimposed on the apparatus of models.

(E) provides a model-theoretic characterization of 'formality': to be formal is not to distinguish between (be invariant under) isomorphic structures. It might be worthwhile to trace the origins of this criterion: The invariance condition first appeared in the literature (in a somewhat different setting) in Lindenbaum and Tarski's 1935 paper, "On the Limitations of the Means of Expression of Deductive Theories" (see Section II above). A restricted version of (E) is stated in A. Mostowski's 1957 pioneer paper, "On a Generalization of Quantifiers"[21] (with references to the Lindenbaum-Tarski paper and to a 1946 paper by Mautner[22]). Per Lindström[23] introduced the full criterion in his generalization of Mostowski's system (1966). (More accurately, Lindström's definition is limited to relational structures.) Since then, the criterion appeared in the linguistic literature but (as far as I know) the only thorough philosophical discussion of its significance is found in Timothy McCarthy's 1981 paper, "The Idea of a Logical Constant."[24] McCarthy rejected (E) as a criterion for 'logicality' on the grounds that it does not prevent the definition of logical terms by means of 'contingent' expressions (i.e., expressions whose extension changes from one 'possible world' to another); as a result, *logical truth* and *logical consequence* fail to be truth preserving. As we saw above, the condition (B) guards against this possibility.

We can now give a semantic definition of Tarskian logical terms:

(TLT) C is a *Tarskian logical term* iff C is a truth-functional connective or C satisfies the conditions (A)–(E) above on logical constants.

We will call logical terms of the types (C)(a) and (C)(b) above *logical predicates* and *logical functors*, respectively. Logical terms of type (C)(c) we will call *logical quantifiers*, and logical terms of type (C)(d)—*logical quantifier-functors*.

What kind of expressions satisfy (TLT)? Clearly, all the logical constants of standard 1st-order logic do. Identity and the standard quantifiers can be defined by total functions on models, f_I, $f_\forall$ and $f_\exists$, such that, given a model $\mathcal{A}$ with a universe A:

(10) $f_I(\mathcal{A}) = \{<a,b>: a,b \epsilon A \ \& \ a=b\}$.
(11) $f_\forall(\mathcal{A}) = \{B: B=A\}$.
(12) $f_\exists(\mathcal{A}) = \{B: B \subseteq A \ \& \ B \neq \phi\}$.

The definitions of the truth-functional connectives remain unchanged. Among the nonstandard terms satisfying (TLT) are the following:

(13) The 'cardinal' quantifiers (1-place 'predicative' logical quantifiers, i.e., quantifiers whose arguments are 1-place 1st-order predicates), defined by: $f_\alpha(\mathcal{A}) = \{B: B \subseteq A \ \& \ |B| = \alpha\}$, where α is any cardinal number and '$|B|$' stands for 'the cardinality of B'.
(14) The 1-place predicative quantifiers 'finitely many' and 'uncountably many', defined by: $f_{finite}(\mathcal{A}) = \{B: B \subseteq A \ \& \ |B| < \aleph_0\}$ and $f_{uncountably\text{-}many}(\mathcal{A}) = \{B: B \subseteq A \ \& \ |B| > \aleph_0\}$, respectively.
(15) The 1-place predicative quantifier 'as many are . . . as are not', defined by: $f_{as\text{-}many...}(\mathcal{A}) = \{B: B \subseteq A \ \& \ |B| = |A-B|\}$.
(16) The 1-place predicative quantifier 'most' (as in 'Most things are B's'), defined by: $f^1_M(\mathcal{A}) = \{B: B \subseteq A \ \& \ |B| > |A-B|\}$.
(17) The 2-place predicative quantifier 'most' (as in 'Most B's are C's'), defined by: $f^2_M(\mathcal{A}) = \{<B,C>: B,C \subseteq A \ \& \ |B \cap C| > |B-C|\}$.
(18) The 'well-ordering' quantifier (a 1-place, 'relational' quantifier

over 2-place relations), defined by:
$f_{WO}(\mathscr{A})$ = {R: $R \subseteq A^2$ & R is a strict linear ordering such that every nonempty subset of Fld(R) has a minimal element in R}.

(19) The 'ordering-of-the-natural-numbers-with-0' quantifier (a 2-place relational quantifier over pairs of a 2-place relation and an individual), defined by:
$f_{>N,0}(\mathscr{A}) = \{<R,a>: R \subseteq A^2$ & $a \epsilon A$ & and $<A,R,a>$ is a model of the natural numbers with their ordering relation and zero}.

(20) 'The first' logical functors (n-place, for any n), defined by: $f_{first}(\mathscr{A})$ = the function $g:A^n - > A$, such that for any n-tuple $<a_1,\ldots,a_n> \epsilon A^n$, $g(a_1,\ldots,a_n) = a_1$.

(21) The 1-place 'complement' quantifier-functor, defined by: $f_{complement}(\mathscr{A})$ = the function $g:P(A) - > P(A)$ such that for any $B \subseteq A$, $g(B) = A - B$.

Examples of constants which do not satisfy (TLT):

(22) The 1-place predicate 'identity with *a*' (*a* is an individual constant of the language), defined by: $f_{=a}(\mathscr{A}) = \{b: b \epsilon A$ & $b = a^{\mathscr{A}}\}$, where $a^{\mathscr{A}}$ is the denotation of *a* in $\mathscr{A}$.

(23) The 1-place (predicative) 'pebbles in the Red Sea' quantifier, defined by: $f_{pebbles}(\mathscr{A})$ = {B: $B \subseteq A$ & B is a nonempty set of pebbles in the Red Sea}.

(24) The 1st-order membership relation, defined by:
$f_\epsilon(\mathscr{A}) = \{<a,b>: a,b \epsilon A$ & b is a set & a is a member of b}.

The definitions of these constants violate (E). (To see why (24) fails, think of two models $\mathscr{A}$ and $\mathscr{A}'$ with universes $\{\phi,\{\phi\}\}$ and {Jean-Paul Sartre, Albert Camus}, respectively.)

Another term which is not logical under (TLT) is the definite-description operator, ℩ . If we define ℩ (a quantifier-functor) by a function f which, given a model $\mathscr{A}$ with a universe A, assigns to $\mathscr{A}$ a partial function h from P(A) into A, then (C)(d) is violated. If we make h universal, using some convention to define the value of h for subsets of A which are not singletons, the convention is likely to violate (E). We can, however, construct a 2-place predicative logical quantifier, 'The', which expresses Russell's contextual definition of the description operator:

(25) $f_{The}(\mathscr{A}) = \{<B,C>: B \subseteq C \subseteq A$ & B is a singleton set}.

We can add any collection of Tarskian logical terms to a standard system of 1st-order logic by providing appropriate syntactic and semantic definitions. E.g., the 2-place 'most' could be introduced, syntactically, as a binary formula-building operator, defined by: If Φ and Ψ are well-formed formulas, then $(\text{most}^2\ x)[\Phi,\Psi]$ is a well-formed formula. Its inductive definition of satisfaction in a model could be formulated as follows:

> Given a model $\mathscr{A}$ with a universe A and an assignment g of individuals in A to the variables of the language:
> If Φ, Ψ are well-formed formulas, then $\mathscr{A} \models (\text{most}^2\ x)[\Phi,\Psi][g]$ iff $< \{a \in A : \mathscr{A} \models \Phi[g(x/a)]\}, \{a \in A : \mathscr{A} \models \Psi[g(x/a)]\} > \in f^2_M(\mathscr{A})$
> (where '$\mathscr{A} \models \Phi[g]$' means: 'g satisfies Φ in $\mathscr{A}$', and $[g(x/a)]$ is an assignment which assigns a to x and otherwise is the same as g).

As can be seen from this example, the syntactic and semantic definitions of logical terms under (TLT) are essentially Tarskian. We will call a standard 1st-order logic with additional Tarskian logical terms and with syntax and semantics as indicated above a *generalized 1st-order system*. We can now define the notion of *Tarskian Logic*:

(TL) $\mathscr{L}$ is a *1st-order Tarskian Logic* iff $\mathscr{L}$ is either a standard or a generalized 1st-order system.

We claim that (TL) satisfies Tarski's pretheoretical requirements. Namely, if $\mathscr{L}$ is a 1st-order Tarskian logic, then the Tarskian definition of 'logical consequence' for $\mathscr{L}$ gives results in accordance with (C1) and (C2).

(C1): We will show that the assumption (AS) holds for $\mathscr{L}$. (As we saw in Section II above, this is sufficient to prove that (C1) holds for $\mathscr{L}$.) Let $\mathscr{C}$ be the logical vocabulary of $\mathscr{L}$ and L—its extra-logical vocabulary. The strong semantic variability of terms in L ensures that every possible state of affairs relative to L is represented by some model $\mathscr{A}$ for $\mathscr{L}$. We have to show that the same holds for $L \cup \mathscr{C}$. Claim: if Φ is a well-formed formula of $\mathscr{L}$, every possible extension of Φ relative to the vocabulary of $\mathscr{L}$ is represented by some model for $\mathscr{L}$ (where the extension of a sentence is taken to be a truth value, T or F).

I will sketch an outline of a proof. Suppose Φ is an atomic formula of the form 'Px', where P is an extra-logical constant. Then, since $P \in L$, the claim holds for Φ. Suppose Φ is of the form '$(Qx)\Psi x$', where Q is a quantifier, and 'Ψx' is (for the sake of simplicity) a formula with one free variable, x. Assume the claim holds for 'Ψx'. Q is a logical constant whose meaning is rigid: i.e., the mathematical interpretation of Q and its intuitive meaning coincide. Therefore, it is intuitively possible for '$(Qx)\Psi x$' to have the extension T/F iff it is possible that 'Ψx' has the kind of extension which can be represented by a subset B of the universe of some model $\mathscr{A}$ for $\mathscr{L}$ such that $B \in f_Q(\mathscr{A})/B \notin f_Q(\mathscr{A})$. By assumption, every (relevant) possible extension of 'Ψx' is represented by some model for $\mathscr{L}$.

So, if it is possible for 'Ψx' to have an extension as above, there is a model which realizes this possibility. In this model the extension of '$(Qx)\Psi x$' is T/F. (It is easy to extend this argument to the more general case in which 'Ψx' has other free variables besides x.) We can carry on this inductive reasoning with respect to any type of logical terms under (TLT).

(C2): Condition (E) expresses a notion of formality based on Tarski's criterion. To be formal is, semantically, to take only structure into account. Within the scheme of model-theoretic semantics this means to be invariant under isomorphic structures. Now languages without logical constants cannot yield (nontrivial) logical consequences. (By a trivial consequence of a set K of sentences I mean a sentence X such that $X \epsilon K$.) Therefore, logical consequences are due to logical constants which, satisfying (E), are formal constituents of the language.

V. A New Conception of Logic

Our discussion so far has shown that (i) a distinction between logical and extra-logical terms is essential for Tarskian logic in view of its purpose, (ii) there exists an independent criterion for logical terms satisfying Tarski's requirements, and (iii) the list of logical terms under this criterion far exceeds that of standard logic.

The answer to the question posed at the beginning of this paper is now clear: The basic semantic principles of Tarskian logic are not exhausted by the standard system. The logical consequences of standard 1st-order mathematical logic are not all the necessary-and-formal consequences of 1st-order languages (i.e., languages whose variables and nonlogical constants are of the first order). It takes the complete range of Tarskian Logics—TL—to fully realize Tarski's program.

Given the prominent place of standard 1st-order logic in contemporary philosophy, mathematics, and related disciplines, the question naturally arises whether a revision in the 'official' doctrine of logic is called for. At stake is a change in a very general and basic conceptual scheme, and the pros and cons of revisions of this kind touch upon a variety of issues. In the remainder of this paper I would like to dwell, if very briefly, upon some of these.

REVISION IN LOGIC. Regarding the conditions under which conceptual revision is justified, Putnam convincingly argued that a change in a deeply ingrained conceptual scheme is seriously entertainable only if a well-developed alternative has already been developed.[25]

Is there a serious alternative to standard logical theory incorporating the principles of Tarskian Logics delineated above? The answer is: Yes, there exists a rich body of literature, in mathematics as well as in linguistics, in which nonstandard systems of 1st-order logic satisfying (TL) have been developed, studied, and applied. The first to propose a 'generalized' 1st-order logic was Andrzej Mostowski in his 1957 paper (see above). Mostowski constructed a system of logical quantifiers defined by cardinality functions like those in examples (13)–(16) above. A generalization of Mostowski's system was proposed by Per Lindström (see above) who identified 1st-order quantifiers with classes of 1st-order relational structures (of a given type) closed under isomorphism. All our examples but (19) fall under Lindström's definition. Among the most interesting mathematical results concerning generalized systems of 1st-order logic is a completeness theorem due to J. Keisler[26] who proved that a 1st-order system with the quantifier 'there are uncountably many' and a modest set of logical axioms is complete. The applicability of nonstandard quantifiers to natural-language semantics was first suggested by Jon Barwise and Robin Cooper in their 1981 paper "Generalized Quantifiers and Natural Language."[27] Since then the linguistic scene has featured an abundance of research on nonstandard quantifiers (logical as well as nonlogical, according to our criterion). In particular, a new approach to natural-language determiners is based on the analysis of determiners as generalized quantifiers.[28] Until now no one (to the best of my knowledge) has brought a philosophical argument for the view that generalized logic, in its full scope, is logic proper. If the philosophical analysis proposed in this paper is sound, it adds to the support the new logic received from other quarters.

THE LOGICIST THESIS. The logicist thesis says that mathematics is reducible to logic in the sense that all mathematical theories can be formulated using purely logical means. I.e., all mathematical constants are definable in terms of logical constants and all the theorems of (classical) mathematics are derivable from purely logical axioms using logical rules of derivation (and definitions). Now, for the logicist thesis to be meaningful, the notions of 'logical constant', 'logical axiom', 'logical rule of derivation' and 'definition' must be well-defined and, moreover, be so defined as to make the reduction nontrivial. In particular, it is essential that the reduction of mathematics to logic be carried out relative to a system of logic in which mathematical constants do not, in general, appear as primitive logical terms. The 'fathers' of logicism did not engage in a critical examination of the concept 'logical constant' from this point of view. That is, they took it for granted that there is a small group of constants in terms of which the reduction is to be carried out: the truth-functional connectives, the existential (universal) quantifier, identity, and possibly the set-membership relation. The new conception of logic, however, contests this assumption. If our analysis of the semantic principles underlying modern logic is correct, then any mathematical predicate or functor satisfying (E) can play the role of a primitive logical constant. Since mathematical constants in general satisfy (E) when defined as higher order, the program of reducing mathematics to logic becomes trivial. Indeed, even if the whole of mathematics could be formulated within pure standard 1st-order logic, then (since the standard logical constants are nothing more than certain particular mathematical predicates) all that would have been accomplished is a reduction of some mathematical notions to other mathematical notions.

While the logicist program is meaningless from the point of view of the new conception of logic, its main tenet, that mathematical constants are essentially logical, is, of course, strongly supported by this conception. Indeed, Russell's account of the 'logicality' of mathematics in *Introduction to Mathematical Philosophy* is in complete agreement with our analysis:

> There are words that express form . . . And in every symbolization hitherto invented of mathematical logic there are symbols having constant formal meanings . . . Such words or symbols express what are called 'logical constants'. Logical constants may be defined exactly as we defined forms; in fact, they are in essence the same thing . . . In this sense all the 'constants' that occur in pure mathematics are logical constants.[29]

Aside from 'high-orderization', the main difference between the new conception and Russellian logicism is the direction from which the equation of 'logical' with 'mathematical' via 'formal' is read. While the traditional logicists say that mathematical constants are essentially logical, the new conception claims that logical constants are essentially mathematical. Thus, the 'logical thesis of mathematics' is replaced, in the new conception, by the 'mathematical thesis of logic'.

As for high-orderization, note that this requirement is, in some respects, very Fregean. Frege's *logical* definition of the natural numbers takes numbers to be higher order entities, i.e., classes of classes or classes of concepts. Indeed, the formulation of numerical statements as 1st-order quantifications in Generalized Logic is exactly the same as in Frege's *The Foundations of Arithmetic*.[30]

MATHEMATICS AND LOGIC. Our discussion of logicism above highlighted one aspect of the relationship between logic and mathematics: In Tarskian Logics any mathematical constant can play the role of a logical term subject to certain requirements on its syntactic and semantic definitions. However, mathematical constants appear in Tarskian Logics also as extra-logical constants, and this reflects another side of the relationship between logic and mathematics: As logical terms, mathematical constants are constituents of logical frameworks in which theories of various kinds are formulated and their logical consequences are drawn. But the 'pool' of formal terms which can figure as logical constants in Tarskian Logics is created in mathematics. The semantic definition of, say, the logical quantifier 'there are uncountably many x' is based on some mathematical theory of sets. Similarly, the semantic definitions of the truth-functional connectives and the universal (existential) quantifier are based on certain simple Boolean algebras. These observations point to a difference between logic and mathematics vis-à-vis formal terms: Formal terms are *generated* in mathematics, while *used* in logic.

Now, since logic provides a framework for theories in general, the meanings of formal terms can be given by mathematical theories formulated within logic. We can thus picture the interplay between logic and mathematics as a cumulative process of definition and application. Starting with a logical system which applies certain elementary, but powerful mathematical functions (Boolean truth-functions, the universal/existential-quantifier function and, usually, identity), to a 1st-order extra-logical vocabulary, we construct various formal theories. Such theories describe mathematical structures using extra-logical vocabulary which is undefined in the language, but which receives specific meaning by the theorems of the theory. Once mathematical (extra-logical) terms are defined by theories within the framework of standard 1st-order logic, they can be used to create a super-structure for a new, extended, system of logic. As an example consider the 1st-order theory of Peano Arithmetic. The arithmetic terms defined within this theory can be converted into logical arithmetic quantifiers, to be included in the super-structure of a new, extended system of logic. We will then use the new logical framework to formulate theories—mathematical, physical, etc.—which assume the existence of a machinery for counting and comparing sizes. In these theories we will conclude *logically* that, say, there are 4 B's, given that there are 2 C's and that the number of B's is twice the number of C's. As we shall see below, there is an essential difference between applying mathematics by using mathematical terms as part of the logical super-structure and applying mathematics by adding extra-logical mathematical constants and axioms to a theory of standard 1st-order logic.

LOGIC AND ONTOLOGY. Quine is known for the thesis that the logical structure of theories in a standard 1st-order formalization reflects their ontological commitments. To determine the ontology of a theory $\mathcal{T}$ formulated in natural language (or a scientific 'dialect' thereof) we formalize it as a (standard) 1st-order theory, $\mathcal{T}1$, and examine those models of $\mathcal{T}1$ in which the extra-logical terms receive their intended meaning. $\mathcal{T}$ is committed to the existence of such objects as populate the universes of the intended models. Thus, if $\mathcal{T}$ includes a sentence of the form:

(26) Uncountably many things have the property P,

then, since the notion of uncountably many is not definable in pure standard 1st-order logic, we have to expand $\mathcal{T}1$ by combining it with another theory, for example, some set theory with urelements, in which 'uncountably many' can be defined. We then formalize (26) as:

(27) $(\exists x)$[x is a set & x is uncountable & $(\forall y)(y \epsilon x \rightarrow$ y is an individual & Py)].

Through (27) $\mathcal{T}$ is committed to the existence of sets.

Now, consider what happens if we formalize $\mathcal{T}$ by a theory $\mathcal{T}2$ of a Tarskian logic, $\mathcal{L}$, obtained from standard 1st-order logic by adding the logical quantifier 'uncountably many' together with appropriate axioms (e.g., Keisler's). Obviously, we do not need set theory in order to express (26) in $\mathcal{L}$. The meaning of (26) is adequately captured by the sentence.

(28) (Uncountably many x) Px,

which does not commit $\mathcal{T}$ to the existence of sets. So, with a 'right' choice of logical vocabulary, $\mathcal{T}$ can make do with an ontology of mere individuals.

We see that the new conception of logic allows us to save on ontology by augmenting the logic. We can weaken the ontological commitments of theories by parsing more terms as logical. We no longer talk about *the* ontological commitment of a non-formal theory (there is no such thing!); instead ontological considerations become a factor in choosing logical frameworks for formalizing theories.

The examination of Quine's principle from the perspective of Tarskian Logics throws light on the crucial role played by logical constants in deciding ontological commitments. Indeed, logical constants are central to ontological commitment on other theories of logic and ontology as well. Consider the simple, straightforward view that the commitment of a theory $\mathcal{T}$ under a formalization $\mathcal{F}_{\mathcal{T}}$ is determined by what is common to all models of $\mathcal{F}_{\mathcal{T}}$. Here, too, the difference in logical terms between the formalizations $\mathcal{T}1$ and $\mathcal{T}2$ of $\mathcal{T}$ results in essentially different commitments. The occurrence of the quantifier 'uncountably many' in (28) ensures that in every model of $\mathcal{T}2$ P is assigned an uncountable set of individuals. But, by the Skolem-Löwenheim theorem, $\mathcal{T}1$ has at least one model in which the predicate 'x is uncountable' is given a nonstandard interpretation and P is assigned a countable set. We can thus say that $\mathcal{T}2$ is committed to an ontology of uncountably many objects, whereas $\mathcal{T}1$ is not.

We see that logical terms are vehicles of *strong ontological commitment*, while extra-logical terms transmit a relatively weak commitment. This difference in ontological import between logical and extra-logical terms is explained by the fact that logical terms are, semantically, pre-fixed, whereas the meaning of extra-logical terms is relative to models. In Putnam's form of speech, extra-logical terms are viewed from *within* models, logical terms are viewed from the *outside*.[31] Including formal terms as part of the logical super-structure allows us to use them in the logic with 'a view from the outside'.

This distinction provides the explanation (promised earlier) for the difference between using mathematical notions as part of the logical 'machinery' and using them as extra-logical terms of theories within the logical framework. It also provides a guideline for choosing logical frameworks: If you formulate, say, a physical theory, and you want to use formal tools created elsewhere (i.e., in some mathematical theory), you might as well include the mathematical apparatus as part of the logical super-structure. This will reflect the fact that you are not interested in specifying the meanings of the mathematical terms but in saying something about the physical world using mathematical notions which you take as given. The pre-fixed notions will enable you to make some very strong claims about the physical world, strong in the sense that what they say does not vary from one model of the theory to another. All this will be done without compromising the usefulness of the logical framework in determining necessary-and-formal consequences. If, on the other hand, your goal is to define the mathematical notions themselves, you cannot construe them as logical because as such their meaning has to be given to begin with. You have to use undefined terms of the language (i.e., extra-logical terms) and then construct a theory within which these notions will receive a distinctive content.[32]

PROOF THEORETICAL PERSPECTIVE. The philosophical justification of the new conception of logic is based on an analysis of certain *semantic* principles underlying Tarskian logic. What about proof theory? Does the new conception satisfy the main principles of the modern theory of proof? One may be tempted to issue a quick verdict: The new logic fails to satisfy the most important requirement of proof theory—a complete proof procedure. I think this might be a premature judgment. The 'new conception of Tarskian logic' is a result of re-examining the philosophical ideas behind logical semantics in response to certain mathematical generalizations of standard semantic notions (Mostowski et al.). There is no sense in comparing the generalized semantics with current, un- or pre-generalized, proof theory. To do justice to the new conception from a proof-theoretic perspective one has to cast a new, critical look, at the standard notion of proof. This task may be exacting because (as far as I know) there is no rich body of mathematical generalizations in proof theory parallel to 'generalized logic' in contemporary model theory.

However, if the work in generalized semantics is philosophically significant, it poses a challenge to proof-theory that cannot be overlooked. We can put it this way: If Tarski is right about the basic intuitions underlying our conception of logical truth and consequence, and if our analysis is correct—namely, these intuitions are not exhausted by standard 1st-order semantics—then, since standard 1st-order logic is complete, these intuitions are not exhausted by standard (1st-order) proof-theory either. Semantically, we have seen, it suffices to enrich the super-structure of 1st-order logic by adding new logical terms. But what has to be done proof-theoretically?

VI. Postscript: Tarski and the New Conception of Logic

What would Tarski have thought about the conception of 'Tarskian' logic proposed in this paper? After the first few versions of the paper had been completed I came upon a 1966 lecture by Tarski (first published in 1986) which provides a partial answer to our question. In this lecture, titled "What are Logical Notions?",[33] Tarski proposed a definition of 'logical term' which is a restriction of our condition (E) to structures within a given model:

> Consider the class of *all* one-one transformations of the space, or universe of discourse, or 'world' onto itself. What will be the science which deals with the notions invariant under this widest class of transformations? Here we will have . . . notions all of a very general character. I suggest that they are the logical notions, that we call a notion 'logical' if it is invariant under all possible one-one transformations of the world onto itself.[34]

Tarski concluded that no singular term is logical, but any predicate definable in standard higher order logic is. Thus, as a science of individuals mathematics is different from logic, but as a science of higher-order structures mathematics *is* logic.

The main difference between Tarski's lecture and the present paper has to do with the grounds for the extension. Tarski introduced his conception as a generalization of Klein's classification of geometrical disciplines according to the transformations of space under which the geometrical concepts are invariant. Abstracting from Klein, Tarski characterized logic as the science of all notions invariant under 1–1 transformations of the universe of discourse ('space' in a generalized sense). My own conclusions, on the other hand, were based on the analysis of Tarski's early views on the (philosophical) foundations of semantics and, in light of these, of 'how models work'. I will end this paper by saying that what I called the 'new' conception of Tarskian logic is not really new: Tarski himself arrived at essentially the same conception, although based on a different analysis from the one proposed here.

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NOTES

[1] *Synthese* 31 (1975), pp. 1-21.

[2] Ibid., p. 5.

[3] Several interesting papers are devoted to the issue of the scope of logic from points of view which are to a greater or lesser degree similar to the one taken here. Unfortunately, however, due to the length of the present work, I will be able to relate only to those views which are directly connected to my ideas. Among the said papers are, in addition to Tharp, C. Peacocke, "What Is a Logical Constant?", *Journal of Philosophy* 73 (1976), pp. 221-240, I. Hacking, "What Is Logic?", *Journal of Philosophy* 76 (1979), pp. 285-319, T. McCarthy, "The Idea of a Logical Constant", *Journal of Philosophy* 78 (1981), pp. 499-523, J. Etchemendy, "The Doctrine of Logic as Form", *Linguistics and Philosophy* 6 (1983), pp. 319-334, G. Boolos, "To Be Is to Be a Value of a Variable (or to Be Some Values of Some Variables)", *Journal of Philosophy* 81 (1984) pp. 430-449, and D. Westerstahl, "Logical Constants in Quantifier Languages", *Linguistics and Philosophy* 8 (1985), pp. 387-413.

[4] The Polish original of "The Concept of Truth in Formalized Language" dates from 1933. "On The Concept of Logical Consequence" was first printed in Polish in 1936. Both papers (in English translation by J. H. Woodger) appear in Alfred Tarski, *Logic, Semantics, Metamathematics*, 2nd ed.; ed. and introduced by J. Corcoran (Hackett Publishing Company, 1983), pp. 152-278 and 409-420, respectively. See also "The Establishment of Scientific Semantics", there, pp. 401-408.

[5] *Proceedings of the Tarski Symposium*, ed. L. Henkin, et al. (Providence: The American Mathematical Society, 1974), pp. 153-172.

[6] Ibid., p. 161.

[7] See articles III-VI, VIII-X, XII and XIV in *Logic, Semantics, Metamathematics*, especially pp. 30, 36-37, 38-40, 60-63, 69-72, 166, 281, 285, 298 and 342.

[8] See "On The Concept of Logical Consequence", pp. 410-412 and references there.

[9] Ibid., p. 412.

[10] "The Establishment of Scientific Semantics", p. 401.

[11] "On The Concept of Logical Consequence", pp. 414-415.

[12] Ibid., p. 417.

[13] Ibid.

[14] This argument appears in an unpublished manuscript based on Etchemendy's Ph.D Thesis, "Tarski, Model Theory, and Logical Truth" (Stanford University, 1982). Although Etchemendy refers to a notion of model different from that discussed here, the argument does not depend on this notion.

15 "On the Limitations of the Means of Expression of Deductive Theories", *Logic, Semantics, Metamathematics*, pp. 384–392. The citation is from p. 385.

16 "On a Generalization of Quantifiers", *Fundamenta Mathematicae* 44 (1957), pp. 12–36.

17 "The Concept of Logical Consequence", pp. 418–419.

18 See ibid., p. 420.

19 Ibid., p. 419.

20 Ibid., fn. 1.

21 See fn. 16 above.

22 "An Extension of Klein's Erlanger Program: Logic as Invariant-Theory", *American Journal of Mathematics* 68, pp. 345–384.

23 "First-Order Predicate Logic with Generalized Quantifiers", *Theoria* 32, pp. 186–195.

24 See fn. 3 above.

25 "The Analytic and the Synthetic", *Readings in the Philosophy of Language*, eds. J. F. Rosenberg and C. Travis (Englewood Cliffs, NJ: Prentice-Hall, 1971), pp. 94–126.

26 "Logic with the Quantifier 'There Exist Uncountably Many'", *Annals of Mathematical Logic* 1 (1970), pp. 1–93.

27 *Linguistics and Philosophy* 4 (1981), pp. 159–219.

28 In addition to Barwise & Cooper, see J. van Benthem, *Essays in Logical Semantics* (Dordrecht: D. Reidel, 1986) and references there.

29 2nd ed. (London: Allen and Unwin, 1920), p. 201.

30 Tr. by J. L. Austin (Evanston, Ill.: Northwestern U Press, 1968).

31 See "The Thesis That Mathematics Is Logic", *Mathematics: Matter and Methods—Philosophical Papers*, Vol. 1, 2nd ed. (Cambridge, England: Cambridge University Press, 1979), p. 16.

32 Some of the themes developed in this section regarding the interplay between logic and ontology appear earlier in Charles Parsons' "Ontology and Mathematics" (1971) and "A Plea for Substitutional Quantification" (1971), *Mathematics in Philosophy: Selected Essays* (Ithaca, N.Y.: Cornell University Press, 1983), pp. 37–62 and 63–70, respectively. The observation that by augmenting one's logic one can save on ontology was made earlier by Hartry Field in *Science Without Numbers: A Defense of Nominalism* (Princeton University Press, 1980), Preface and Chapter 9.

33 Ed. and introduced by J. Corcoran, *History and Philosophy of Logic* 7 (1986), pp. 143–154.

34 "What Are Logical Notions?" p. 149.